SINGULAR DYNAMICS
IN
QUANTUM MECHANICS AND QUANTUM FIELD THEORY

By

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Dedicated to

My Parents and Friends

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The classical viewpoint that the pathologies of quantum Hamiltonians are reflected in their classical counterparts and in the solution of the associated classical dynamics is illustrated through the properties of classical equations of motion. Rigorous mathematical proofs are provided by calculating the deficiency indices, which further help to analyze the possibility of extending the Hermitian Hamiltonian to a self-adjoint one. A few examples of how to add boundary conditions on wave functions in order to have self-adjoint extensions, as well as to cure the illness of certain Hamiltonians, are discussed.

The nontrivial (non-Gaussian), nonperturbative quantum field theory of $O(N)$-invariant multi-component, nonrenormalizable ultralocal models is presented. An indefinite nonclassical and singular potential has replaced the nonvanishing, positive-definite nonclassical and singular potential appeared in the single-component case. The operator theory of multi-component ultralocal

fields remains noncanonical even with the disappearance of the singular potential. The validity of this nontrivial solution is supported by the fact that the nontrivial quantum solution reduces to the correct classical theory in a suitable limit as $\hbar \to 0$. The nontrivial (non-Gaussian) path integral formulation constructed by the nonperturbative operator solution, involving the nonclassical and singular potential, replaces the standard lattice approach, which invariably leads to a Gaussian theory regardless of any nonlinear interactions. The appearance of the nonclasssical, singular potential suggests that we can not always place the classical Lagrangian or classical Hamiltonian directly into the path-integral formulation, or in other words, a straightforward canonical quantization of fields with infinite degrees of freedom does not always apply. These differences suggest plausible modifications in the lattice-space formulation of relativistic nonrenormalizable models that may lead to nontriviality for N-component models such as Φ^4_n, $n \geq 5$ (and possibly $n = 4$), for $N \geq 1$.

PART I

CLASSICAL SYMPTOMS OF QUANTUM ILLNESSES

CHAPTER 1
INTRODUCTION

The rules of quantization, as laid down in 1926 by Schrodinger, have stood the test of time and have provided the basis for applying (and teaching) nonrelativistic quantum mechanics[1]. Because of the overwhelming success of the theory, it is not surprising that physicists have little use for several mathematical niceties that go by the largely unfamiliar code words of "self-adjoint extension," "deficiency indices," " operator domains," etc. (which we will briefly define in subsequent sections). [2,3]

These concepts typically become important when the quantization prescriptions are ambiguous or otherwise incomplete all by themselves[4]. Our goal is to show that such quantum technicalities are actually reflected in the classical theory by corresponding difficulties that are easy to see and understand. It will be clear that such an evident classical difficulty must lead to some kind of quantum difficulty, and thus the naturalness of the corresponding quantum technicality will become apparent. We may go so far as to assert that there are sufficient classical symptoms of any quantum illness that a complete diagnosis is possible already at the classical level (Chapter 2). Of course, resolving the problems and effecting a full cure, when one exists, can only take place at the quantum level (Chapter 3).

This chapter contains basically a summary of the mathematical concepts and theorems prepared for later chapters. In Chapter 2, we will discuss the properties of classical solutions of given examples, whose "symptoms" can be

used to diagnose any potential quantum "illnesses". In Chapter 3, we will provide the mathematical proof for all the assertions made in Chapter 2 and will also give a few examples of how to add boundary conditions on the wave functions in order to extend the Hamiltonians to be self-adjoint and so to cure the illness of certain Hamiltonians.

1.1 Basic Concepts of Operator Theory

In this section we follow Ref. 3 closely. The operators we deal with in quantum mechanics are frequently unbounded operators which need to be defined on a domain $D(A)$ to ensure that $A\psi \in \mathcal{H}$ for $\psi \in D(A)$, where $\mathcal{H}$ is Hilbert space.

Generally, if A and B are two unbounded linear operators,

$$D(\alpha A + \beta B) = D(A) \cap D(B), \qquad \alpha, \beta \in C,$$
$$(\alpha A + \beta B)\psi = \alpha(A\psi) + \beta(B\psi), \qquad \psi \in D(\alpha A + \beta B).$$

For the product put

$$D(AB) = \{\psi : \psi \in D(B), \quad B\psi \in D(A)\},$$
$$(AB)\psi = A(B\psi), \qquad \psi \in D(AB).$$

In general $D(AB) \neq D(BA)$, and hence $AB \neq BA$, and even $A A^{-1} \neq A^{-1}A$, because their domains may not coincide. So extra care is needed in treating these operators.

We recall that A is densely defined in $\mathcal{H}$ if (and only if) $\mathcal{H}$ is the closure of $D(A)$, i. e., for $\psi \in \mathcal{H}$, there is a sequence of elements of $D(A)$ converting to ψ, $[\phi_n \to \psi, \phi_n \in D(A)]$.

For a densely defined operator A for $\mathcal{H}$ into $\mathcal{H}$, there exists a unique adjoint operator $A^\dagger$ which satisfies

$$(A\,\psi, \phi) = (\psi, A^\dagger \phi),$$

where $\psi \in D(A)$ and $\phi \in D(A^\dagger)$.

If an adjoint operator $A^\dagger = A$ on $D(A)$, and $D(A) = D(A^\dagger)$, then A is called a self-adjoint operator.

1.2 Symmetric Operators and Extensions

If $A^\dagger = A$ on $D(A)$, then $D(A) \subseteq D(A^\dagger)$ always holds (like Hermitian Hamiltonian operators). Now,

(a) A is called a self-adjoint operator when $D(A) = D(A^\dagger)$, and one writes $A = A^\dagger$.

(b) A is called a symmetric operator when $D(A) \subset D(A^\dagger)$, and one writes $A \subset A^\dagger$.

Further, if there exists a symmetric extension A_s of A, such that $A \subset A_s = A_s{}^\dagger \subset A^\dagger$, then we say that A has the self-adjoint extension A_s (by enlarging $D(A)$ to $D(A_s)$ and contracting $D(A^\dagger)$ to $D(A_s{}^\dagger) = D(A_s)$).

Let $A^\dagger \phi_\pm = \pm i\, \phi_\pm$, with solutions $\phi_\pm \in N_\pm$ which are closed subspaces of $\mathcal{H}$, known as the deficiency spaces of A; and their dimensions, $n_\pm$ respectively, are called the deficiency indices. We shall also say for brevity that A has deficiency indices (n_+, n_-).

Lemma 1. If A is symmetric, then $D(A^\dagger) = D(\overline{A}) \oplus N_+ \oplus N_-$, $\overline{A}$ denotes the closure of A.

Lemma 2 A symmetric operator with finite deficiency indices has a self-adjoint extension if and only if its deficiency indices are equal.

With the help of the two above Lemmas, the following theorem regarding deficiency indices is straightforward:

THEOREM 1: Suppose the Hermitian operator A has deficiency indices (n_+, n_-), then if

(i) $n_+ = n_- = 0$, A is self-adjoint when $D(A)$ is closed and the closure of A is self-adjoint when $D(A)$ is not closed.

(ii) $n_+ = n_- \neq 0$, A has self-adjoint extensions.

(iii) $n_+ \neq n_-$, A is not self-adjoint and has no self-adjoint extension.

This theorem is used to classify Hermitian operators, and we see that not all Hermitian operators will generate full-time quantum mechanical solutions.

Example: $A = -i\,d/dx$ on $\mathcal{L}_2(\Omega)$

$$A^\dagger = -i\,d/dx\,, \qquad A^\dagger \phi_\pm = \pm i\,\phi_\pm,$$

$$-i\frac{d}{dx}\phi_\pm = \pm i\,\phi_\pm\,, \qquad \phi_\pm(x) \sim exp(\mp x)\,.$$

(a) $\Omega = (-\infty, +\infty)$: Neither ϕ_+ nor ϕ_- is in $\mathcal{L}_2(-\infty, +\infty)$, so $n_+ = n_- = 0$ and A is self-adjoint on its natural domain. This fact is consistent with its classical behavior (see Fig. 1(a)), where there is a full-time evolution.

(b) $\Omega = [\,0, +\infty)$: $\phi_- = \exp(x)$ is not in $\mathcal{L}_2([\,0, +\infty))$, so $(n_+, n_-) = (1, 0)$, A is not self-adjoint and possesses no self-adjoint extension. This fact is consistent with its classical behavior (see Fig. 1(b)), where there is no full-time evolution.

(c) $\Omega = [\,0, 1]$: Both $\phi_\pm$ are in $\mathcal{L}_2([\,0, 1\,])$, $(n_+, n_-) = (1, 1)$, A has self-adjoint extensions by adding the boundary condition $\psi(1) = \exp(i\theta)\,\psi(0)$, where $\theta \in \Re$.

The spectrum of $A = \theta + 2n\,\pi$, $n = 0, \pm 1, \pm 2, \ldots$. For different θ, we get a different self-adjoint extension. The case of $\theta = 0$ is the usual case used in quantum mechanics and is called the periodic boundary condition. See Fig. 1(c) for its consistent classical behavior.

The deficiency indices are clearly of central importance in the classification of Hermitian operators and in the construction of self-adjoint extensions. These may be found easily if explicit solutions of $A^{\dagger}\psi_{\pm} = \pm i\,\psi_{\pm}$ are known, but unless the coefficients of A are simple, the calculations will involve difficult special functions. In order to avoid those tedious, sometimes even impossible calculations, we will use a modified theorem of deficiency indices, especially when both end points of Ω are singular. First, let us look at an example with quantum Hamiltonian H, where

$$H\,\psi = (p^2 + x^3)\psi = (-\frac{d^2}{dx^2} + x^3)\psi$$

acting on functions defined on an interval $\Omega = (-\infty, +\infty)$. Both end points $x = \pm\infty$ are singular. A direct calculation of the deficiency indices seems incredibly tedious since the behavior of a solution of $H\psi_{\pm} = \pm i\,\psi_{\pm}$ at both end points must be known. To simplify the difficulties, we separate $\Omega = (a, b)$, where a and b are both singular, to two intervals with $\Omega_1 = (a, c\,]$ and $\Omega_2 = [\,c, b)$, where c is a regular point. Here comes the second theorem regarding deficiency indices:

THEOREM 2 Let A_a and A_b be symmetric operators associated with A on Ω_1 and Ω_2, respectively. If $n_{\pm}$, $n_{\pm a}$, $n_{\pm b}$ indicate the deficiency indices of A, A_a and A_b, respectively, then $n_{\pm} = n_{\pm a} + n_{\pm b} - 2$, respectively (see Chapter 2 of Ref. 3).

To apply this result to the above example, we have

$$H^{\dagger}\psi_{\pm} = (-\frac{d^2}{dx^2} + x^3)\psi_{\pm} = \pm i\,\psi_{\pm}$$

Since H is a real operator, i. e., $H^* = H$, we have $\psi_- = \psi_+^*$. So $n_+ = n_-$ and we only need to consider ψ_+ and n_+. Let $\psi_+ = \psi$,

$$(-\frac{d^2}{dx^2} + x^3)\psi = i\psi.$$

The asymptotic behaviors at $x \to \pm\infty$ is given by

$$\psi'' - x^3\psi = 0$$

$$\psi_{x\to+\infty} \sim \sqrt{x}\,K_{1/5}(\frac{2}{5}x^{5/2}) \sim x^{-3/4}\exp(-\frac{2}{5}x^{5/2})$$

$$\psi^{(1)}_{x\to-\infty} \sim \sqrt{|x|}\,J_{1/5}(\frac{2}{5}|x|^{5/2}) \sim |x|^{-3/4}\cos(\frac{2}{5}|x|^{5/2} - \frac{7}{20}\pi)$$

or

$$\psi^{(2)}_{x\to-\infty} \sim \sqrt{|x|}\,N_{1/5}(\frac{2}{5}|x|^{5/2}) \sim |x|^{-3/4}\sin(\frac{2}{5}|x|^{5/2} - \frac{7}{20}\pi)$$

where J_ν, N_ν and K_ν are the Bessel functions of the first and second kinds, respectively. Obviously, the integration

$$\int|\psi(x)|^2dx \qquad \text{converges at } x = +\infty.$$

Thus, $n_{+,+\infty} = 1$. In the same way,

$$\psi^{(1,2)}_{x\to-\infty} \sim |x|^{-3/4}$$

so that both solutions of $\psi^{(1,2)}_{x\to-\infty}$ satisfy

$$\int|\psi(x)|^2dx \qquad \text{converges at } x = -\infty.$$

Therefore, $n_{+,-\infty}= 2$, and

$$n_+= n_{+,+\infty} + n_{+,-\infty} - 2 = 1 + 2 - 2 = 1$$

$$(n_+, n_-) = (1, 1).$$

One boundary condition* is needed to extend H to be self-adjoint. The existence of bound states $\psi_{\pm\infty}$ demonstrates that the energy spectrum is discrete. This result is exactly what is claimed in Chapter 2.

Later in Chapter 3, we will see examples for three different cases of the deficiency indices: (i) $n_+= n_- = 0$, (ii) $n_+= n_- \neq 0$, and (iii) $n_+\neq n_-$.

1.3 The Construction of Self-Adjoint Extensions

Before we construct the self-adjoint extensions, we introduce some notation (see Chapter 10 of Ref. 3):

(i) $\langle\phi, \psi\rangle = (A^\dagger\phi, \psi) - (\phi, A^\dagger\psi)$, where $\phi, \psi \in D(A^\dagger)$;

(ii) $f_1, f_2 \cdots f_n \in D(A^\dagger)= D(\overline{A}) \oplus N_+\oplus N_-$, where $f_i = h_i + \phi_i$, $i = 1, 2, \ldots,$ $h_i \in D(\overline{A})$, $\phi_i \in N_+\oplus N_-$.

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* If $n_+= n_-$, the number of boundary conditions = n_+; see page 260 of Ref 3.

We call f_1, $f_2 \cdots f_n$ linearly independent relative to $D(A)$ if $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent.

With the help of the preceding definitions, we introduce the following theorem:

THEOREM 3 Let A be a symmetric operator with finite nonzero deficiency indices $n_+ = n_- = n$. Suppose that $f_1, f_2 \cdots f_n \in D(A^\dagger)$ are linearly independent relative to $D(A)$ and satisfy

$$\langle f_i, f_j \rangle = 0 \qquad (i, j = 1, 2, \ldots, n).$$

Then the subspace $\mathcal{M}$ of $\mathcal{H}$ consisting of all $\psi \in D(A^\dagger)$ such that $\langle \psi, f_i \rangle = 0$ ($i = 1, 2,$ ...., n) is the domain of a self-adjoint extension M of A, given by $M\psi = A^\dagger\psi$ for $\psi \in \mathcal{M}$.

$\langle \psi, f_i \rangle = 0$, $i = 1, 2, ..., n$, are called the number of n boundary conditions on the wave function ψ.

In chapter 3, we will give a few examples of how to use theorem 3 to obtain some of these self-adjoint extensions and thereby cure the illness of certain Hamiltonians.

CHAPTER 2
CLASSICAL SYMPTOMS OF QUANTUM ILLNESSES

Without using the precise definitions and techniques of operator theory, presented in chapter 1, we can simply show that the pathologies of quantum Hamiltonians are reflected in their classical counterparts and in the solutions of the associated classical equations of motion. This property permits one, on the one hand, to appreciate the role of certain technical requirements acceptable quantum Hamiltonian must satisfy, and, on the other hand, enables one to recognize potentially troublesome quantum systems merely by examining the classical systems.

In §2.1, we will make the principal assertions that three categories of classical solutions correspond to three categories of the self-adjointness of quantum Hamiltonian operators. Examples are given in §2.2 and §2.3, while the mathematical proof will be presented in chapter 3.

2.1 Principal Assertions

Specifically, for the classical system with a real Hamiltonian function $H = \frac{p^2}{2m} + V(q)$, the classical equation of motion

$$m\ddot{q}(t) = - V'[q(t)]$$

admits two kinds of solution:

(i) $q(t)$ is global (which is defined through $-\infty < t < +\infty$) and unique, e. g.,

$H = p^2 + q^4$;

(ii) $q(t)$ is locally unique, but globally possibly nonexistent (escapes to infinity in finite time). This case may be also divided into two sub-cases: extendible ($q(t)$ can be extended to a global solution) or nonextendible. For example, $H = p^2 - q^4$, $H = p^2 \pm q^3$ are extendible, but $H = pq^3$ belongs to nonextendible situation.

The corresponding quantum mechanical problem

$$i\hbar\frac{\partial}{\partial t}\psi(x, t) = H\psi(x, t) = [\, -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\,]\ \psi(x, t)$$

has similar cases. To conserve probability, the time evolution of the wave function must be effected by a unitary transformation. A unitary transformation gives a prescription for wave function evolution for all times t, $-\infty < t < \infty$, or as we shall sometimes also say, for full time. The generator of such a unitary transformation times $\sqrt{-1}$, which we identify with the Hamiltonian operator (up to constants), must satisfy one fundamental property, namely, that of being "self-adjoint." Being self-adjoint is stronger, i.e., more restrictive, than being Hermitian, which is the generally accepted sufficient criterion. According to Theorem 1 on page 5, typically, there are three qualitatively different outcomes that may arise (i) no additional input is needed to make the Hamiltonian self-adjoint; (ii) some additional input (i.e., the boundary conditions) is required to make the Hamiltonian self-adjoint, and qualitatively different Hamiltonians may thereby emerge depending on what choice is make for the needed input; and (iii) no amount of additional input can ever make the Hermitian Hamiltonian into a self-adjoint operator. Remember it is only self-adjoint Hamiltonians that have fully consistent dynamical solutions for all time. Thus in case (iii) no acceptable quantum dynamical solution exists in any conventional sense.

Therefore we make the general assertions that there are three categories of classical solutions, which correspond to three categories of the self-adjointness of quantum Hamiltonian operators:

	$q(t)$	Hamiltonian operator
(i)	global & unique	self-adjoint & unique
(ii)	nonglobal but extendible	self-adjoint extensions exist
(iii)	nonglobal & nonextendible	no self-adjoint H exists

2.2 Classical Symptoms of Quantum Illnesses

To be sure, the majority of classical systems one normally encounters--as represented by their classical Hamiltonians, their Hamiltonian equations of motion, and the solution to these equations of motion--are trouble free. But there are exceptions, classical systems which exhibit one or another kind of singularity in their solutions, and it is in these problematic examples that we will find the anomalous behavior we seek to illustrate.

Let us consider three model classical Hamiltonians that will exhibit the three kinds of behavior alluded to above.

Example a:

$$H = \frac{1}{2}p^2 + \frac{1}{4}q^4.$$

The equations of motion lead to

$$\dot{q}^2 = 2E - \frac{1}{4}q^4,$$

where $E \geq 0$ is the conserved energy. Nontrivial motion requires $E > 0$ and $E \geq q^4/4$ in which case

$$\pm \int^{q(t)} \frac{dq}{\sqrt{2E - \frac{1}{4}q^4}} = t + c$$

leads to a well-defined, unique solution valid for all time. This solution exhibits a periodic behavior with an energy dependent period $T(E)$. See example a in Table I.

Example b:

$$H = \frac{1}{2}p^2 + \frac{1}{3}q^3.$$

The equations of motion lead to

$$\dot{q}^2 = 2E - \frac{2}{3}q^3$$

and E can assume any real value. Since $q^3 \leq 3E$, we find that

$$\pm \int^{q(t)} \frac{dq}{\sqrt{2E - \frac{2}{3}q^3}} = t + c.$$

This solution has the property that q diverges, $q \rightarrow -\infty$, for finite times, for any nonzero E; see the solid line of example *b* in the table. The indicator of this behavior is the observation that the integral converges as the upper limit goes to $-\infty$ leading to a finite value for the right side, namely, a finite value for the time. Actually, this same trajectory diverges at an earlier time as well. As *b* illustrates,

the particle comes in from $q = -\infty$ at some time, say $t = t_0$, and returns to $q = -\infty$ at a later time, say $T + t_0$, where

$$T = 2\int_{-\infty}^{(3E)^{1/3}} \frac{dq}{\sqrt{2E - \frac{2}{3}q^3}}.$$

For any nonzero E, $T < \infty$; in particular, by a change of variables, we learn that

$$T = \frac{2^{1/2}3^{1/3}}{|E|^{1/6}}\int_{-S(E)}^{\infty} \frac{dx}{\sqrt{1 + x^3}},$$

where $S(E) = E/|E|$, the sign of the energy. As $|E| \to 0$ the particle spends more and more time near $q = 0$, until, at $E = 0$, it takes an infinite amount of time to reach (or leave) the origin.

Example *b* in the table illustrates the generic situation for $E \neq 0$ and finite T. How could one possibly expect the quantum theory to persist for all time when the generic classical solution diverges at finite times? The only possible way for this example to have a genuine quantum mechanics is for the particle to enjoy a full-time classical solution. And to achieve that--and this is the important point--whenever the particle reaches $q = -\infty$, we must launch the particle back toward the origin with the same energy, once again follow the trajectory inward, and then outward, until the particle again reaches $q = -\infty$, when we must again launch the particle back toward the origin with the same energy, ..., and so on, both forward and backward in time *ad infinitum.* In brief, to get full-time classical solutions (as needed to parallel full time quantum solutions) we must recycle the

same divergent trajectory over and over again in an endless periodic fashion (with period $T(E)$); see *b* in the table with the solid and the dashed lines.

Example c:

$$H = p\, q^{\,3}.$$

The equations of motion, in this case, lead to

$$\dot{q} = q^3,$$

with the solution

$$q(t) = \pm\, 1/\sqrt{c - 2t}.$$

The solution is valid for $t < c/2$, diverges at $t = c/2$, and becomes *imaginary* for $t > c/2$. See example *c* in the table, where we chose the constant $c = 0$. In no way can an imaginary solution be acceptable as a classical trajectory. We cannot relaunch the particle once it has reached $q = \pm\, \infty$ at $t = c/2$ following the kind of solution we have found, the only solution there is, as we were able to do in case *b*. In short there is no possibility to have a full-time classical solution in the present example.

If there is no full-time classical solution, then there should be no full-time quantum solution, and that is exactly what happens. The corresponding quantum Hamiltonian may be chosen as Hermitian, but there are no technical tricks that can ever make it self-adjoint. Once again, the signal of this quantum behavior can be seen in the classical theory, i.e., a divergence of the classical

solution followed by a change of that classical solution from real to imaginary. With such a classical symptom, it is no wonder that there is an incurable quantum illness.

2.3 More General Examples

In what follows we discuss a number of hypothetical classical Hamiltonians, examine qualitatively the nature of their classical solutions, and address, based on the thesis illustrated in Sec. 2.1, the problem of making an Hermitian Hamiltonian into a self-adjoint one, if that is indeed possible.

Let us dispense with the "healthy" cases at the outset. Whenever the classical equations of motion admit *global* solution for *arbitrary* initial conditions, then the quantization procedure is unambiguous (apart from classically unresolvable factor-ordering ambiguities). Observe that the existence of such global solutions is an intrinsic property of a Hamiltonian independent of any particular set of canonical coordinates.

2.3.1. Examples of One-Dimensional Hamiltonians

Example d:

$$H = \frac{1}{2}p^2 - q^4,$$

$$\dot{q} = p = \pm \sqrt{2}\sqrt{E + q^4},$$

$$\int^{q(t)} \frac{dq}{\sqrt{2}\sqrt{E + q^4}} = \pm t + c.$$

E can be any real value.

For $E > 0$, we notice that $\int_{-\infty}^{+\infty} dq \,/\, \sqrt{2}\sqrt{E + q^4}$ converges to, say $T(E)$, which means when the particle travels from $-\infty$ to $+\infty$, the time interval is $T(E)$. For nonzero E, T is finite; see the solid line of example d in the table. In order to get a full-time classical behavior, we must send the particle back when it gets $q = \pm\infty$; see the two distinct dashed lines of d in the table. In quantum theory, this situation corresponds to adding two boundary conditions (at $q = \pm\infty$), which are required to extend H to be self-adjoint.

For $E < 0$, the paths do not cross the origin, but there is no essential difference in the quantum behavior from the case $E > 0$. That means we still need to add two boundary conditions to extend H to be self-adjoint.

Example e:

$$H = \frac{1}{2}p^2 - q^2,$$

$$\dot{q} = p = \pm\sqrt{2}\sqrt{E + q^2},$$

$$\int^{q(t)} \frac{dq}{\sqrt{2}\sqrt{E + q^2}} = \pm t + c,$$

$$(1/\sqrt{2}) \ln \left| q + \sqrt{q^2 + E} \right| = \pm t + c.$$

When the particle travels from $-\infty$ to $+\infty$, the time spent by the particle spans the whole t-axis (see example e in the table), so there is a global solution for an

arbitrary initial condition. Thus the quantum Hamiltonian is unique and self-adjoint.

Since there is no periodic behavior, the spectrum of the quantum Hamiltonian is continuous.

Example f:

$$H = 2\,p\,q^{2},$$

$$\dot{q} = 2q^{2},$$

$$2q = 1/(-\,t + c).$$

Global solutions exist (see example *f* in the table) and the quantum Hamiltonian is self-adjoint. Since there is no periodic behavior the spectrum is continuous.

Example g:

$$H = 2\,p\,(1/q),$$

consider $\Omega = (-\infty, 0)$ or $(0, +\infty)$,

$$\dot{q} = 2(1/q),$$

$$q^{2} = 4t + c.$$

similar to Example *c*, q becomes imaginary for $t < -c/4$, and there is no possibility of having a full-time classical solution, just as in the case of example *c*. Therefore there is no way to extend the Hamiltonian to be self-adjoint.

Example h:

$$H = p^2 + 2pq^3,$$

$$\dot{q} = 2(p + q^3) = \pm 2\sqrt{q^6 + E}.$$

Its classical solution is similar to example *d*, and is identical to $H = p^2 - q^6$, to which it is canonically equivalent.

Example i :

$$H = p^2 + 2p/q$$

$q \neq 0$, so $\Omega = (-\infty, 0)$ or $(0, +\infty)$,

$$\dot{q} = 2(p + 1/q) = \pm 2\sqrt{E + 1/q^2},$$

$$\sqrt{1 + Eq^2} = \pm 2E(t + c).$$

The solution to the motion reads

$$(2E)^2(t + c)^2 - Eq^2 = 1.$$

See example *i* in the table, where we chose $E > 0$. Since $q = 0$ is a singular point, we divide $\Re$ into $\Omega = (-\infty, 0)$ and $(0, +\infty)$. For example, when the particle reaches $q \to 0^+$, we have to send it back or cross over $q = 0$ to get a full-time classical behavior. By choosing *c*, the full-time classical picture may be like the solid and dashed lines of *i* in the table. In quantum theory, one boundary condition is needed to extend *H* to be self-adjoint.

Two additional examples (*l* and *m*) appear in the table without discussion in the text.

2.3.2. Examples of Three-Dimensional Hamiltonians

To simplify the problem, we consider here only central potentials $V(r)$. In suitable units, the quantum Hamiltonian becomes

$$\widehat{H} = -\nabla^2 + V(r),$$

$$\nabla^2 = \frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{\widehat{L}^2}{r^2}.$$

Set

$$\psi(\mathbf{x}) = \frac{u(r)}{r} Y_{lm},$$

then

$$H_r = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r) \qquad \text{for } u(r).$$

The condition that $\int |\psi|^2 \, d^3r$ converges now becomes the condition that $\int_0^\infty |u(r)|^2 dr$ converges.

For $V(r) = -\lambda / r^n$, $n = 1,2,3,\ldots$, $\lambda > 0$,

$$H_r = p_r^2 - (\lambda / r^n) + (l\,(l+1)/r^2)$$

and the classical equation of motion is

$$\dot{r} = 2\sqrt{E + (\lambda / r^n) - (l\,(l+1)/r^2)}.$$

Under both cases of $E < 0$ and $E > 0$, we see the difference between $n = 1$ and $n \geq 2$. For $n \geq 2$, near the origin, i.e., $r \sim 0$, $\dot{r} \approx 2\sqrt{E + \lambda / r^n}$, so r could go as near as r ~ 0. See example k in the table. In quantum theory, one condition is required to extend the Hamiltonians to be self-adjoint. It corresponds to one parameter (e.g., see parameter B in Ref. 4), which is needed to specify quantum solutions. But for $n=1$, $\dot{r} = 2\sqrt{E + \lambda / r - (l+1)/r^2}$; with $E < 0$ and $l > 0$, the motion oscillates between the two roots of the expression under the radical, namely,

$$+\frac{\lambda}{2|E|} - \sqrt{\frac{\lambda^2}{4E^2} - \frac{l(l+1)}{|E|}} \leq r \leq +\frac{\lambda}{2|E|} + \sqrt{\frac{\lambda^2}{4E^2} - \frac{l(l+1)}{|E|}}.$$

So it avoids the singular point $r = 0$, and therefore it has a full-time classical evolution (see example j in the table), thus its quantum Hamiltonian is a self-adjoint operator. That is why the hydrogen atom does not have any problems.

CHAPTER 3
SELF-ADJOINTNESS OF HERMITIAN HAMILTONIANS

Having recalled that simple and physically natural classical "symptoms" are available to diagnose any potential quantum "illness," we now discuss some of the standard techniques in Chapter 1 used to analyze Hamiltonian operators and confirm that the classical viewpoint presented in Chapter 2 is fully in agreement with conventional analyses.

In particular, we analyze in this chapter the self-adjointness of several Hamiltonians which appeared in Chapter 2 by calculating their deficiency indices directly. Therefore, we can verify the consistency of solutions of classical and quantum Hamiltonians. According to Theorem 1 on page 5, we have three different cases of the deficiency indices: (i) $n_+ = n_- = 0$, (ii) $n_+ = n_- \neq 0$, and (iii) $n_+ \neq n_-$. Cases (i) and (iii) are simple, where either a densely defined operator is or is not self-adjoint, but case (ii) is much more complicated. Discussions such as how to add boundary conditions on the wave functions in order to extend the Hamiltonians to be self-adjoint, and what kind of physical interpretation those extensions imply, etc., will be presented in §3.2 and §3.3.

3.1 Three Different Cases of Hermitian Hamiltonians

In this section we will apply *Theorem s1 & 2* to four general types of Hamiltonian, just as we did in §1.2 for $H = p^2 + x^3$:

(1) $H = p^2 - bx^m$,

(2) $H = p\,x^m + x^m\,p$,

(3) $H = p^2 + p\,x^m + x^m\,p$,

(4) $H = p^2 + x\,p^m + p^m\,x$

where m is a positive integer and b is a real constant. Those four types of H cover all the Hamiltonians discussed in chapter 2.

3.1.1 $H = p^2 - bx^m$

Obviously, H is Hermitian. Similarly to what we did for $H = p^2 + x^3$ in §1.2, let

$$H^\dagger \psi_\pm = \pm i\,\psi_\pm$$

with $p = -\,i\,d\,/dx$,

$$\left(\frac{d^2}{dx^2} + bx^m\right)\psi_\pm = \mp i\psi_\pm.$$

As both b and m are real, $\psi_- = \psi_+^*$. Thus, $n_+ = n_- = n$. So consider ψ_+ only; we write

$$\left(\frac{d^2}{dx^2} + bx^m\right)\psi = -\,i\psi.$$

The asymptotic behavior of this equation as $x \to \pm\infty$ is given by

$$\psi'' + bx^m\psi = 0$$

since we require $\psi(\infty) = 0$. The solutions are given by Bessel functions:

$$\psi(x) \sim \sqrt{|x|}\, Z_{1/(m+2)}\left(\frac{2\sqrt{b}}{m+2}\sqrt{x^{m+2}}\right)$$

where $Z_\nu \sim J_\nu$, N_ν, I_ν, or K_ν, which are Bessel functions of the first and second kinds, respectively. To examine whether or not $\int |\psi(x)|^2 dx$ converges at $x \to \infty$, we need to use the following asymptotic forms of the Bessel functions:

$|y| \to +\infty$,

$$J_\nu(y) \sim \sqrt{\frac{2}{\pi y}}\cos\left[y - (\nu+\frac{1}{2})\frac{\pi}{2}\right]$$

$$N_\nu(y) \sim \sqrt{\frac{2}{\pi y}}\sin\left[y - (\nu+\frac{1}{2})\frac{\pi}{2}\right]$$

$$I_\nu(y) \sim \frac{1}{\sqrt{2\pi y}}\exp(y)$$

$$K_\nu(y) \sim \sqrt{\frac{\pi}{2y}}\exp(-y)$$

If $\int |\psi(x)|^2 dx$ converges, then $\psi_\pm(x) \in D(H^\dagger) \subset \mathcal{H}$.

Examples:

(a) $H = p^2 + x^4$:

$$\psi'' - x^4\psi = -i\psi$$

near $|x| \sim \infty$, $\psi \to 0$; thus,

$$\psi'' - x^4\psi = 0.$$

Then $b = -1$ and $m = 4$. We should choose $Z \sim I_\nu, K_\nu$ (with $y = 1/3\ |x|^3$), the Bessel functions of the second kind.

For $x \sim \pm\infty$, $\psi(x) \sim \sqrt{|x|}K_{1/6}(\frac{1}{3}|x|^3)$,* so $n_{+,\pm\infty} = 1$

$n_+ = n_{+,+\infty} + n_{+,-\infty} - 2 = 1 + 1 - 2 = 0 = n_-$, i. e., $(n_+, n_-) = (0, 0)$.

Thus, H is self-adjoint in $\mathcal{L}_2((-\infty, +\infty))$, which makes possible the global solution of the Schrodinger evolution equation (see Sections 9.6 and 9.7 of Ref. 3); the existence of the bound states ψ_+ and ψ_- implies a discrete spectrum of energy.

(b) $H = p^2 - x^4$:

$$\psi'' + x^4\psi = -i\psi$$

Here $b = +1$ and $m = 4$; thus

$$\psi_{+\infty}^{(1,2)}(x) \sim \sqrt{x}J_{1/6}(\frac{1}{3}x^3) \quad \text{and} \quad \sqrt{x}N_{1/6}(\frac{1}{3}x^3) \sim (1/x)\cos \text{ or } \sin(\frac{1}{3}x^3 - \frac{1}{3}\pi).$$

It is easy to check that both of the solutions are in $\mathcal{L}_2$, so $n_{+,+\infty} = 2 = n_{+,-\infty}$ as ψ is symmetric on x.

$n_+ = 2 + 2 - 2 = 2 = n_-$, i.e., $(n_+, n_-) = (2, 2)$.

Two boundary conditions are required to extend H to be self-adjoint, and a discrete energy spectrum is expected.

Indeed, even a WKB analysis leads to a discrete spectrum.

---

* I_ν will make the integral diverge, and therefore is not allowed.

(c) $H = p^2 - x^2$ ($p^2 + x^2$ is similar to $p^2 + x^4$)

$$\psi'' + x^2\psi = - i\psi$$

where $b = 1$ and $m = 2$.

$$\psi_{\pm\infty}^{(1,\,2)}(x) \sim \sqrt{|x|}J_{1/4}(\frac{1}{4}x^2) \text{ and } \sqrt{|x|}N_{1/4}(\frac{1}{4}x^2) \sim |x|^{-1/2}\cos \text{ or } \sin(\frac{1}{4}x^4 - \frac{3}{8}\pi)$$

but neither of them are in $\mathcal{L}_2(\Omega)$; therefore,

$$(n_+, n_-) = (0, 0)$$

and H is self-adjoint, but its spectrum is continuous because no bound states exist. Again, this conclusion follows from WKB arguments as well.

(d) $H = p^2 + x$ ($p^2 - x$ is basically the same)

$$\psi_+'' - x\psi_+ = - i\psi_+$$

$$\psi_+'' - (x - i)\psi_+ = 0.$$

For $x > 0$, $b = -1$ and $m = 1$,

$$\psi_+(x) \sim \sqrt{x - i}\,K_{1/3}(\frac{2}{3}(x - i)^{3/2}),$$

which is in $\mathcal{L}_2$.

But for $x < 0$, $b = +1$, and $m = 1$,

$$\psi_{+}^{(1,2)}(x) \sim \sqrt{|x| + i}\, J_{1/3}(\tfrac{2}{3}(|x| + i)^{3/2}), \quad \sqrt{|x| + i}\, N_{1/3}(\tfrac{2}{3}(|x| + i)^{3/2}) \sim 1/|x|^{1/4}$$

where we have ignored the unimportant phase term. Both of the solutions are not in $\mathcal{L}_2$ (the square integral of ψ_+ diverges at $x \sim -\infty$). That means there is no solution for $x < 0$, and no solution could be connected to ψ_+, $x > 0$ either. Therefore, we have

(i) $(n_+, n_-) = (0, 0)$ for $\Omega = (-\infty, 0)$,

(ii) $(n_+, n_-) = (0, 0)$ for $\Omega = (-\infty, +\infty)$,

(iii) $(n_+, n_-) = (1, 1)$ for $\Omega = [0, +\infty)$.

The spectrum is expected to be continuous in (i) and (ii), but discrete in (iii).

This is an interesting system. In section 3.2, we apply it to a particle in a gravitational field and a uniform electrical field.

3.1.2 $H = p\,x^m + x^m\,p$

By substituting $p \sim -i\,d/dx$ and using $[\,p, x^m\,] = -i\,m\,x^{m-1}$, we get

$$(p\,x^m + x^m\,p)\psi_{\pm} = (2p\,x^m - imx^{m-1})\psi_{\pm} = \pm i\,\psi_{\pm}$$

$$2x^m\psi_{\pm}' = (\mp 1 - mx^{m-1})\psi_{\pm}$$

(i) $m = 1$, $2x\psi_{\pm}' = (\mp 1 - 1)\psi_{\pm}$,

$$2x\psi_{+}' = -2\psi_{+}, \qquad x\psi_{-}' = 0$$

with $\psi_+ \sim 1/x.$, which is not in $\mathcal{L}_2$ since it diverges too rapidly at $x \sim 0$, and $\psi_- =$ const., which is not in $\mathcal{L}_2$ either. Therefore,

$$(n_+, n_-) = (0, 0)$$

Thus, $H = p\,x\, + x\,p$ is self-adjoint and its spectrum is continuous.

(ii) $m > 1$,

$$2\psi_\pm'/\psi_\pm = (\mp 1 - mx^{m-1})/x^m$$

$$\psi_\pm \sim |x|^{-m/2}\exp\left(\pm \frac{1}{2(m-1)}x^{1-m}\right)$$

As $x \sim \pm\infty$, $\psi_\pm \sim 1/|x|^{m/2}$, which are square integrable, so $\pm\infty$ are regular points. But now $x = 0$ is a singular point. Checking $\psi_\pm$ near $x = 0$, we find

m = odd, $n_- = 1$, $n_+ = 0$, which has no self-adjoint extension. H is not self-adjoint and has no complete evolution, just as was the case for $H = p\,x^3\, + x^3\,p$ in Chapter 2.

m = even, $n_- = n_+ = 0$, H is self-adjoint with continuous spectrum. See the example of $H = p\,x^2\, + x^2\,p$ in the previous chapter.

It is interesting to point out that for $\Omega = (-\infty, 0)$, $(n_+, n_-) = (0, 1)$ if m = odd and (1, 0) if m = even; for $\Omega = (0, +\infty)$, $(n_+, n_-) = (0, 1)$ if m = odd and (0, 1) if m = even. Therefore, H is not self-adjoint in both $\Omega = (-\infty, 0)$ and $\Omega = (0, +\infty)$, no matter whether m is even or odd. To explain that by the classical picture (see *e*xample *c* in the table), we say that neither of those intervals could contain a complete temporal evolution.

3.1.3 $H = p^2 + p x^m + x^m p$

In the above section, we found that $p x^m + x^m p$ is not self-adjoint if m = odd. Here we will show that by adding p^2 to it, the situation will be changed dramatically. First let us examine the classical solutions:*

$$H = p^2 + p q^m + q^m p = p^2 + 2p q^m$$

$$\dot{q} = 2p + 2q^m = \pm 2\sqrt{E + q^{2m}}$$

(i) If m = 1, the solution is like that for $H = p^2 - q^2$, where H is self-adjoint.

(ii) If m > 1, the solution is similar to that for $H = p^2 - q^4$, where boundary conditions must be added to make H self-adjoint.

Second, let us see that quantum features of H:

$$H = p^2 + px^m + x^m p = p^2 + 2p x^m - imx^{m-1}$$

$$(p^2 + 2p x^m - imx^{m-1})\psi_\pm = \pm i \psi_\pm$$

$$\psi_\pm'' + 2i x^m \psi_\pm' + imx^{m-1}\psi_\pm = \mp i\psi_\pm$$

The asymptotic for near $x \sim \pm\infty$ is

$$\psi_\pm'' + 2i x^m \psi_\pm' + imx^{m-1}\psi_\pm = 0.$$

---

* $H = p^2 + 2p q^m, 2p = -2q^m \pm 2\sqrt{E + q^{2m}}$, $2\dot{p} + 2p^m = \pm 2\sqrt{E + q^{2m}}$

We introduce an equation with the solution given by the Bessel functions, i.e.,

$$\Phi'' + [(1 - 2\alpha)/x \mp 2i\beta\gamma x^{\gamma-1}]\Phi' + [(\alpha^2 - \nu^2\gamma^2)/x^2 \mp i\beta\gamma(\gamma - 2\alpha)x^{\gamma-2}]\,\Phi = 0$$

$$\Phi = x^{\alpha} exp\,(\pm i\beta x^{\gamma})\, Z_{\nu}(\beta x^{\gamma}).$$

Note that $\psi_{\pm}$ just fits this equation by choosing

$$\alpha = 1/2, \qquad \gamma = m + 1, \qquad \beta = 1/\gamma, \qquad \nu = 1/2(m + 1),$$

$$\psi_{\pm\infty} \sim \sqrt{|x|}\exp\left(-i\frac{1}{m+1}x^{m+1}\right) Z_{\nu}\left(\frac{1}{m+1}x^{m+1}\right) \sim 1/\sqrt{|x|^{m}}$$

which is square integrable when $m > 1$*. So $n_{+,\pm\infty} = n_{-,\pm\infty} = 2$, and

$$(n_+, n_-) = (2, 2), \qquad \text{when } m > 1.$$

So by adding p^2, H always has self-adjoint extensions no matter whether m is even or odd. This is exactly consistent with what we have just discussed from the classical point of view.

As for the case $m = 1$, $\psi_{\pm} \sim 1/\sqrt{|x|}$, which is not square integrable, so $(n_+, n_-) = (0, 0)$, which again agrees with what follows from the classical point of view.

3.1.4 $H = p^2 + x^m p + p x^m$

Classical method:

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* Here $Z_{\nu} \sim J_n, N_{\nu}$.

$$\dot{p} = - \partial H/\partial q = - 2\, p^{m}$$

which is similar to $\dot{q} = 2\, q^{m}$ based on $H = 2\, p\, q^{m}$; therefore,

when m = even, H is self-adjoint;

when m = odd, H is not self-adjoint.

Quantum method:

$$x \rightarrow i\, d/dp, \qquad [x,\ p^{m}] = imp^{\,m-1}$$

$$(p^2 + x\, p^{m} + p^{m} x\,)\psi_{\pm}(p) = \pm i\, \psi_{\pm}(p)$$

$$2ip^{m}\psi_{\pm}' = (\pm i - p^2 - imp^{\,m-1})\psi_{\pm}$$

$$\psi_{\pm} \sim (1/\sqrt{|p|^{m}})\, \exp\left(\mp \frac{1}{2(m-1)}\, p^{1-m}\right) \times \exp\left(- i\, \frac{1}{2(m-3)}\, p^{3-m}\right) \qquad (3.1)$$

For $m > 1$: $(n_{+}, n_{-}) = (0, 0)$, if m = even,

$(n_{+}, n_{-}) = (1, 0)$, if m = odd.

For m = 1, $(n_{+}, n_{-}) = (0, 0)$ which has already been discussed in Section 3.1.3. We see again that the classical picture is consistent with the operator analysis.

It is heuristic to observe that the last factor in Eq. (3.1) comes from the part p^2 in H, and it only induces a phase in the solution, therefore it would not change the convergence or divergence of the square integral, i. e., the self-adjointness of $p^2 + x\, p^{m} + p^{m} x$ is exactly the same as $x\, p^{m} + p^{m} x$, and the latter one turns out to be the same as $p\, x^{m} + x^{m} p$.

In other words, unlike the p^2 in $p^2 + p\, x^{m} + x^{m} p$, the p^2 in $p2 + x\, p^{m} + p^{m} x$ will not change the properties of self-adjointness.

3.2 The Application of Boundary Conditions and Self-Adjoint Extensions at Regular Points

Examples:

(a) $A = p = -i\,d/dx$, with $\Omega = [b, a]$

$$p\psi_{\pm} = \pm i\psi_{\pm}\,, \qquad \psi_{\pm} = \exp(\mp x).$$

Thus, $n_+ = n_- = 1$, apply theorem 3 to this case, we have

$$f_1 = c_1\psi_+ + c_2\psi_-\,, \qquad c_1, c_2 \in C$$

$$0 = \langle f_1, f_1 \rangle = (-i\,df_1/dx,\ f_1) - (f_1, -i\,df_1/dx) = i\,[f_1, f_1^*]\,|_a^b$$

$$= i\,[f_1(b)\,f_1^*(b) - f_1(a)\,f_1^*(a)].$$

Set $f_1(b)/f_1(a) = z$; we have $z\,z^* = 1$, so $z = \exp(i\theta)$, $\theta \in \mathcal{R}$. For self-adjoint $D(M) = [\psi(x):\ \langle \psi,\ f_1 \rangle = 0]$, i. e.,

$$\psi(b)\,f_1^*(b) - \psi(a)\,f_1^*(a) = 0$$

$$\psi(b)/\psi(a) = \exp(i\theta) \qquad (3.2)$$

let us look at the eigenfunctions of p :

$$p\psi_\lambda = \lambda\,\psi_\lambda, \qquad \psi_\lambda \sim \exp(i\lambda x)$$

$$\psi_\lambda(b)/\psi_\lambda(a) = \exp(i\lambda(b-a))$$

ψ_λ has to satisfy (3.2) to be in $D(M)$; so we get

$$\lambda(b-a) = \theta + 2\pi n \quad , \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\lambda = (\theta + 2\pi n)/(b-a).$$

The case of $\theta = 0$ is what is usually considered in quantum mechanics, and is called the periodic boundary condition.

(b) $H = p^2 + x$, with $\Omega = [0, +\infty)$

As we already discussed in (d) of Section 3.1.1, $n_+ = n_- = 1$ for $\Omega = [0, +\infty)$. Therefore*

$$0 = \langle f_1, f_1 \rangle = (p^2 f_1 + x f_1, f_1) - (f_1, p^2 f_1 + x f_1) = (p^2 f_1, f_1) - (f_1, p^2 f_1)$$

$$= -(f_1 f_1^{*\prime} - f_1' f_1^*)\Big|_0^{+\infty} = f_1'(0) f_1^*(0) - f_1(0) f_1^{*\prime}(0).$$

So

$$f_1(0)/f_1'(0) = \left[f_1(0)/f_1'(0)\right]^* = \tan\theta \qquad \theta \in \Re,$$

$D(M) = [\psi(x): \langle \psi, f_1 \rangle = 0]$, so the boundary condition at $x = 0$ is

$$\psi(0)\cos\theta - \psi'(0)\sin\theta = 0 \qquad (3.3)$$

---

* $\lim_{x \to \infty} [f_1' f_1^* - f_1 f_1^{*\prime}] = 0$, since when $x \sim +\infty$, $\psi_+ = \psi_-^* \sim \sqrt{x - i}\, K_{1/3}(\frac{2}{3}(x-i)^{3/2})$, $f_1 = c_1\psi_+ + c_2\psi_-$.

In fact, at any regular point, the form is the same.

Consider a particle with 1/2 unit mass moving vertically in the earth's gravitational field and set $x = 0$ at the surface of the earth, $\Omega = [0, +\infty)$. Then its Hamiltonian is given by

$$H = p^2 + mgx = p^2 + x, \qquad \text{by setting } g = 2.$$

It is interesting to note that this system has a natural self-adjoint extension set by y(0) = 0. That is the case of q = 0 in the above boundary condition (3.3).

Additionally consider a physical system of an electron in a uniform electric field E applied in the $+x$ direction. The form of the Hamiltonian is then $H = p^2 + x$ with $\Omega = (-\infty, +\infty)$. For such a system, no special boundary condition is needed since $n_+ = n_- = 0$.

3.3 The Application of Boundary Conditions and Self-Adjoint Extensions at Singular Points

In this section, we will look at the effect of an attractive central potential. We discussed in the classical point of view that there is no trouble for systems with nonrelativistic Coulomb potentials, since they have a full time evolution. Here we will see that this holds because they are self-adjoint. We will also find self-adjoint extensions for the super-attractive potentials [$V(r) = -\lambda/r^n$, $n \geq 2$].

After separating the angular part from the radial part, we have

$$H_r = -d^2/dr^2 + l(l+1)/r^2 + V(r)$$

which acts on $u(r)$ with the requirement that $\int_0^\infty |u(r)|^2\, dr$ must converge.

(i) $V(r) = -\lambda/r$, $\lambda > 0$

$$H_r^{\dagger} u_{\pm} = \pm i u_{\pm}, \qquad u_{\pm}'' + (\pm i + \lambda/r - l(l+1)/r^2)u_{\pm} = 0$$

becomes, at small r,

$$u_{\pm}'' + (\lambda/r - l(l+1)/r^2)u_{\pm} = 0$$

$$u_{\pm} \sim \sqrt{r}\, Z_{\pm(2l+1)}(2\sqrt{\lambda r}) \sim \sqrt{r}\,(2\sqrt{\lambda r})^{\pm(2l+1)}.$$

Thus, with $l > 0$, $\qquad n_{\pm, 0^+} = 1.$

At large r, $\qquad u_{\pm}'' \pm i u_{\pm} = 0$

and its is not hard to find that

$$n_{+, +\infty} = 1.$$

Therefore, $(n_+, n_-) = (0, 0)$ for H with a Coulomb potential.

(ii) $V(r) = -\lambda/r^2$

$$u_{\pm}'' + (\pm i + \lambda/r^2 - l(l+1)/r^2)u_{\pm} = 0.$$

At small r, the eigenfunction u_E and $u_{\pm}$ have the same form:

$$u_{\pm}'' + \frac{\lambda - l(l+1)}{r^2} u_{\pm} = 0$$

$$u_{\pm},\ u_E \sim r^{\rho}, \qquad \rho = 1/2 \pm \sqrt{1/4 - (\lambda - l(l+1))}$$

For an s-wave ($l = 0$) with $\lambda > 1/4$ or $\lambda - l(l+1) > 1/4$, we have

$$u_{\pm},\ u_E \sim r^{1/2}\sin(\lambda' \ln r), \quad \text{or} \quad r^{1/2}\cos(\lambda' \ln r)$$

where $\lambda' = [\lambda - l(l+1) - 1/4]^{1/2}$. Then $n_{\pm, 0^+} = 2$. With $n_{\pm, +\infty} = 1$, we get $(n_+, n_-) = (1, 1)$.

Now let us construct self-adjoint extensions: at small r,

$$f_1 = c_1 r^{1/2}\cos(\lambda' \ln r) + c_2 r^{1/2}\sin(\lambda' \ln r).$$

Using $\langle \phi, \psi \rangle_{0^+} = lim_{r \to 0^+} [\phi' \psi^* - \phi \psi^{*'}]$, and noting that $\langle\ ,\ \rangle$ is a bilinear form on $D(H^\dagger) \times D(H^\dagger)$, and $\langle g, h \rangle = -\langle h, g \rangle^*$, we have

$$\langle r^{1/2}\sin(\lambda' \ln r), r^{1/2}\sin(\lambda' \ln r) \rangle_{0^+} = 0$$

$$\langle r^{1/2}\cos(\lambda' \ln r), r^{1/2}\cos(\lambda' \ln r) \rangle_{0^+} = 0$$

$$\langle r^{1/2}\cos(\lambda' \ln r), r^{1/2}\sin(\lambda' \ln r) \rangle_{0^+} = \lambda'$$

Thus,

$$0 = \langle f_1, f_1 \rangle = (c_1^* c_2 - c_1 c_2^*)\lambda', \qquad c_1^* c_2 = \text{real}.$$

Consequently c_2/c_1 is also real. Set $c_2/c_1 = \tan\theta$, where $\theta \in \Re$; then

$$f_1 \sim r^{1/2}\cos(\lambda' \ln r - \theta)$$

The eigenfunctions have the same form at small r, i. e.,

$$u_E = C_E r^{1/2}\cos(\lambda' \ln r - \theta)$$

in order to satisfy $\langle u_E, f_1 \rangle = 0$. In Ref. 4, the author found the dependence of eigenenergy on the choice of θ. Because of θ, we have a one-parameter family of self-adjoint extensions.

Further, $D(M) = [\, u(r): \langle u, f_1 \rangle = 0\,]$, and therefore the boundary condition at $r = 0$ is given by

$$\lim_{r \to 0^+}\left[u'(r)r^{1/2}\cos(\lambda' \ln r - \theta) - u(r)\ \frac{d}{dr}(r^{1/2}\cos(\lambda' \ln r - \theta))\right] = 0$$

(iii) $V(r) = -\lambda/r^n,\ n > 2,$

$$u_E'' + (E + \lambda/r^n - l(l+1)/r^2)u_E = 0.$$

at small r,

$$u_E'' + (\lambda/r^n)u_E = 0$$

$$u_E \sim \sqrt{r}\, Z_{-1/(n+1)}\left[\frac{-2\sqrt{\lambda}}{n-2} r^{-(n-2)/2}\right]$$

Using the Bessel asymptotic forms, we get

$$u_E \sim r^{n/4}\sin(\gamma r^{-(n-2)/2})$$

or

$$r^{n/4}\cos(\gamma r^{-(n-2)/2})$$

where $\gamma = 2\sqrt{\lambda}/(n-2)$. $u_{\pm}$ have the same form, so $n_{\pm,0^+} = 2$, $(n_+, n_-) = (1, 1)$.

The boundary condition is similar to (ii)

$$f_1 = c_1 r^{n/4}\sin(\gamma r^{-(n-2)/2}) + c_2 r^{n/4}\cos(\gamma r^{-(n-2)/2})$$

using

$$\langle r^{n/4}\sin(\gamma r^{-(n-2)/2}), r^{n/4}\cos(\gamma r^{-(n-2)/2}\rangle_{0^+} = \lambda^{1/2}$$

Thus,

$$\langle f_1, f_1\rangle = 0 \quad \text{gives} \quad c_2/c_1 = \tan\theta, \qquad \theta \in \Re$$

$$f_1 \sim r^{n/4}\sin(\gamma r^{-(n-2)/2} + \theta)$$

$$u_E = C_E\, r^{n/4}\sin(\gamma r^{-(n-2)/2} + \theta)$$

$$D(M) = [\, u(r):\ \langle u, f_1\rangle = 0\,],$$

gives the requirement on u(r) at $r \to 0$:

$$\lim_{r\to 0^+}\left[u'(r)r^{n/4}\sin(\gamma r^{-(n-2)/2} + \theta)) - u(r)\frac{d}{dr}(r^{n/4}\sin(\gamma r^{-(n-2)/2} + \theta))\right] = 0$$

3.4. Conclusion

By calculating the deficiency indices for several representative examples of Hermitian Hamiltonians, we have analyzed the possibilities in each case of extending the operator to a self-adjoint one. In so doing we have confirmed the

connection proposed in the previous chapter that relates possible extensions to associated characteristics in the solutions of the classical equations of motion. The purpose of this chapter has been twofold: on the one hand, by demonstrating that anomalies in the classical solutions to a dynamical system are reflected in anomalies of the quantum Hamiltonian, certain technical issues (such as deficiency indices, etc.) are brought into the realm of everyday experience. On the other hand, the intimate connection between classical and quantum properties should enable one to examine a given system at a classical level in order to assess what problems, if any, are expected to arise at the quantum level.

Table I. Summary of classical and quantum highlights associated with several model problems. Each model has a label, a classical Hamiltonian, typical solution trajectories, nature of those solutions, character of quantum Hamiltonian, spectral properties, and, in some cases, related examples as well. The *solid* curve in the figure portions represents a typical trajectory or part of a trajectory in the case of a periodic orbit. The *dotted* curve denotes an alternative typical trajectory, and the *dashed* curve denotes a periodic extension of the basic orbit. In d, the figure illustrates two possibly distinct periodic extensions. h is omitted because it is a related example of d.

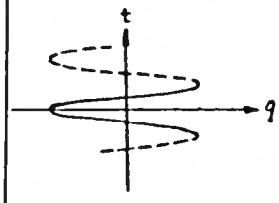
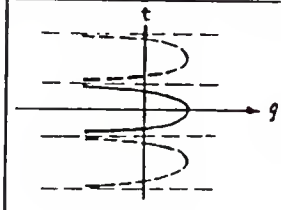
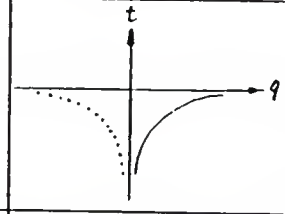
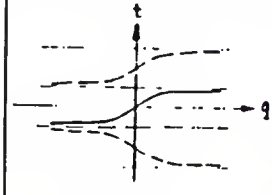
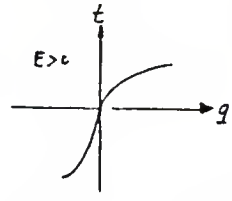
Example Label	Classical Hamiltonian H	Qualitative Graph $t(q)$ of equations of motion	Nature of Classical solutions	Self-adjoint Quantum Hamiltonian	Spectral Properties	Related Examples $m = 1, 2, 3, \cdots$
a	$p^2 + q^4$		global	unique	discrete	$p^2 + q^{2m}$
b	$p^2 + q^3$		periodic	one parameter family of solution (one boundary condition)	discrete	$p^2 - q^3$ $p^2 \pm q^{2m+1}$
c	$2pq^3$		partially complex	nonexistent	none	pq^m , $m > 1$, $m =$ odd
d	$p^2 - q^4$		periodic	two parameter family of solutioo (two boundary conditions)	discrete	$p^2 - q^{2m}$, $m \geq 2$ $p^2 + 2pq^m$, $m \geq 2$
e	$p^2 - q^2$		global	unique	continuous	$p^2 + 2pq$

Table I (*Continued*)

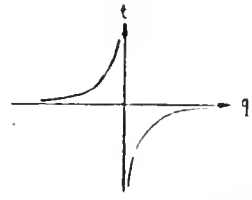
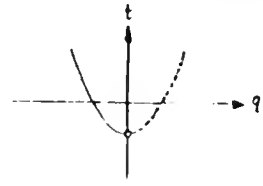
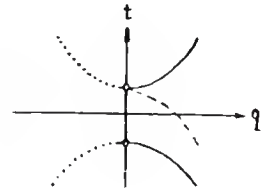
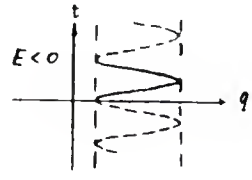
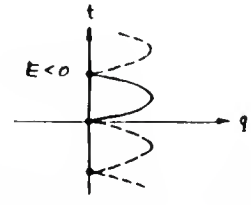
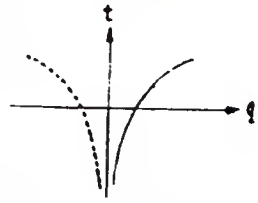
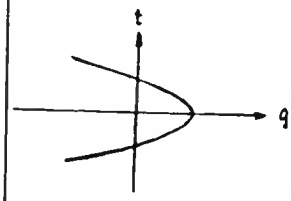
Example Label	Classical Hamiltonian H	Qualitative Graph $t(q)$ of equations of motion	Nature of Classical solutions	Self-adjoint Quantum Hamiltonian	Spectral Properties	Related Examples $m = 1, 2, 3, \ldots$
f	$2pq^2$		global	unique	continuous	pq^m, $m > 1$, m = even
g	$2p/q$		partially complex	nonexistent	none	$2p/q^m$, m = odd
i	$2p/q + p^2$		piece wise continuous	one parameter family of solution	continuous	$p^2 + 2p/q^m$
j	$p_r^2 - \frac{\lambda}{r} + \frac{l(l+1)}{r^2}$ $\lambda > 0$		global	unique	discrete	
k	$p_r^2 - \frac{\lambda}{r^n} + \frac{l(l+1)}{r^2}$		periodic	one parameter family of solution.	discrete	

Table I. (*Continued.*)

Example Label	Classical Hamiltonian H	Qualitative Graph $t(q)$ of equations of motion	Nature of Classical solutions	Self-adjoint Quantum Hamiltonian	Spectral Properties	Related Examples $m = 1, 2, 3, \ldots$
l	$2pq$		global	unique	continuous	
m	$p^2 + q$		global	unique	continuous	

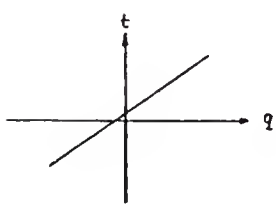
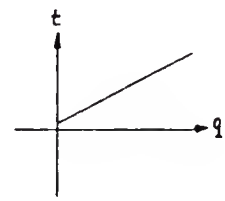
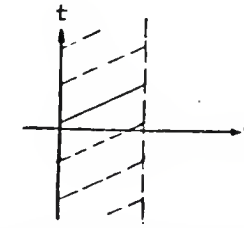




Fig. 1. Classical trajectories for the simple Hamiltonian $H=p$ in three separate coordinate domains: (a) $-\infty < q < \infty$, the trajectory holds for all time; (b) $0<q<\infty$, the trajectory does not hold for all time; (c) $0<q<1$, by periodic continuation of the trajectory it can be extended to hold for all time.

PART II

OPERATOR ANALYSIS AND FUNCTIONAL INTEGRAL REPRESENTATION OF NONRENORMALIZABLE MULTI-COMPONENT ULTRALOCAL MODELS

CHAPTER 4
INTRODUCTION

As we know, the features of infinitely many degrees of freedom and noncompact invariance groups have made the structure of quantum field theory very complex. So seeking for the proper formulation of problems in quantum field theory has been a difficult yet exciting endeavor in our research. Even though conventional perturbation theory[5,6], which is based on a free field formulation, has proved successful in solving some problems, e.g., quantum electrodynamics (QED), we must ascertain whether the free field can be generally used as the basis for a perturbation expansion[7]. One of the results that arose from the traditional perturbation analysis based on free fields is the appearance of nonrenormalizable interactions which some physicists regard as hopeless. Therefore the structure of nonrenormalizable models has been largely ignored over the years. Twenty years ago, the pseudo-free theory was proposed by Klauder[7,8] in the study of ultralocal field models to extend the usefulness of perturbation theory. An argument based on asymptotic convergence suggests that the free theory is connected with continuous perturbations while the pseudo-free theory is related to discontinuous perturbations, which have much to do with nonrenormalizable interactions. At about the same time, a functional and operator approach to single-component ultralocal field models was developed through general nonperturbative and cut-off free arguments[9,10]. There Klauder gave an

alternative quantum theory of such models, which does not fit into the canonical framework. He also showed that as any interaction is turned off. after it was once introduced, the theory will not pass continuously to the free theory, but rather to a distinctly different pseudo-free theory. Clearly, conventional perturbation theory could not be applied to such a model.

The ultralocal model is obtained from covariant model by dropping the space-gradient term. Since distinct spatial points there characterize independent fields for all times, it unavoidably results in a nonrenormalizable situation. We are interested in this model because it is solvable by nonperturbative techniques and may give us some insight into the structure of nonrenormalizable fields.

In chapter 5, we will show that just like nonrenormalizable relativistic quantum field theories, on one hand they exhibit an infinite number of distinct divergences when treated perturbatively, while on the other hand they frequently reduce to (generalized) free fields when defined as the continuum limit of conventional lattice-space formulation[11], ultralocal models are specialized nonrenormalizable theories that also exhibit infinitely many perturbative divergences and an analogous (generalized) free-field behavior when defined through a conventional lattice limit[12]. However, the characterization of infinitely divisible distributions[13] allows ultralocal models alternative operator solutions[9,10] (also Chapter 6), which lead to a nontrivial (non-Gaussian) solution on the basis of operator methods. The validity of this nontrivial solution is supported by the fact that the nontrivial quantum solution reduces to the correct classical theory in a suitable limit as $\hbar \to 0$; the trivial (Gaussian) solution has no such correct classical limit (Chapter 7)[12].

Recently, it has been found how to obtain the same nontrivial results offered by operator techniques through the continuum limit of a

nonconventional lattice-space formulation[14]. The key ingredient in the lattice-space formulation that leads to the correct behavior is the presence of an additional nonclasssical, local potential besides the normally expected terms.

In chapter 6, we will construct the operator theory of $O(N)$-invariant multi-component nonrenormalizable ultralocal models, where $N \leq \infty$. It has been surprisingly found that the singular, nonclassical term in the Hamiltonian which showed up in the one-component ultralocal fields can be made to disappear[15], when N, the number of field components, satisfies $N \geq 4$. Nevertheless, a similar singular, nonclassical term still appears in the regularized path-integral formulation[16] that leads to the same nontrivial quantum results for $O(N)$ invariant multi-component scalar fields (Chapter 8). Thus, for any N, the number of field components, the new path integral formulation involving the singular, nonclassical term replaces the standard lattice approach which invariably leads to a Gaussian theory regardless of any nonlinear interactions, and supports the concept of a pseudo-free theory.

One advantage of deriving the operator solutions by a lattice-space formulation is the clear focus the latter approach places on the differences from traditional approaches needed to lead to nontriviality. These differences for ultralocal models suggest plausible modifications in the lattice-space formulation of relativistic nonrenormalizable modes that may lead to nontriviality for N-component models such as Φ^4_n, $n \geq 5$ (and possibly $n = 4$), for $N \geq 1$.

In my future research, I hope to extend analogous reformulations to covariant quantum fields, the possible relevance of which has already been noted[23]. The general argument is partially based on the realization that the

classical theory of a typical nonrenormalizable interaction exists and is nontrivial[24].

CHAPTER 5
CLASSICAL ULTRALOCAL MODEL AND THE STANDARD LATTICE APPROACH

The classical Hamiltonian of an ultralocal scalar field is expressed as

$$H_{cl} = \int \left\{ \frac{1}{2}\pi^2(x) + \frac{1}{2}m^2\varphi^2(x) + V_I[\varphi(x)] \right\} dx , \qquad (5.1)$$

where π, φ denote the classical momentum and field respectively, and $V_I[\varphi]$ ($= V_I[-\varphi]$, for simplicity) the interaction potential. Here x is a point in configuration space of arbitrary dimension, $x \in \Re^{n-1}$. This model evidently differs from a conventional relativistic field theory by the absence of the term $\frac{1}{2}[\nabla\varphi(x)]^2$. The classical canonical equations of motion appropriate to (5.1) become

$$\dot{\varphi}(x,t) = \frac{\delta H_{cl}}{\delta \pi(x,t)} = \pi(x,t),$$

$$\dot{\pi}(x,t) = -\frac{\delta H_{cl}}{\delta \varphi(x,t)} = -m^2\varphi(x,t) - V_I'[\varphi(x,t)], \qquad (5.2)$$

$$\ddot{\varphi}(x,t) = -m^2\varphi(x,t) - V_I'[\varphi(x,t)].$$

Unlike conventional canonical field quantization, the quantum theory of ultralocal fields does not follow from standard canonical commutation relations, etc., whenever a nonlinear interaction exists. In particular, we do not have an equation such as (5.2) for local quantum operators (see Chap. 6).

Nevertheless, it is instructive to first examine these models from a conventional lattice limit viewpoint.

Let us take a look at the result from a standard functional integral approach (which is based on the standard canonical quantization) where the vacuum-to-vacuum transition amplitude is formally given by [6]:

$$Z[J] = \mathcal{N} \int \mathcal{D}\Phi \exp\left\{ i \int d^4x \left[J(x,t)\Phi(x,t) + \frac{1}{2}\dot{\Phi}^2(x,t) - \frac{1}{2}(m^2 - i\varepsilon)\,\Phi^2(x,t) - g\Phi^4(x,t) \right] \right\},$$

where we have chosen $V_I(\Phi) = g\Phi^4$ as an example. Using a lattice-space regularization in the space direction (but not in the time direction) we obtain

$$Z[J] = \mathcal{N} \lim_{a\to 0} \int \left(\prod_k \mathcal{D}\Phi_k\right) \exp\left\{ i \int dt \sum_k a \left[J_k(t)\Phi_k(t) + \frac{1}{2}\dot{\Phi}_k^2(t) - \frac{1}{2}(m^2 - i\varepsilon)\,\Phi_k^2(t) - g\Phi_k^4(t) \right] \right\}$$

$$= \mathcal{N} \lim_{a\to 0} \prod_k \int \mathcal{D}\Phi \exp\left\{ i \int dt\, a \left[J_k(t)\Phi(t) + \frac{1}{2}\dot{\Phi}^2(t) - \frac{1}{2}(m^2 - i\varepsilon)\,\Phi^2(t) - g\Phi^4(t) \right] \right\}$$

where k labels points in the spatial lattice and a is the volume of the unit cell in the spatial lattice. As we now show, the result for such an interacting model leads in the continuum limit to the analog of a generalized free field. From the preceding equation, using $u = \Phi\sqrt{a}$, $g_0 = g/a$, and $< \cdot >$ as an average in the complex distribution, we have

$$Z[J] = \mathcal{N} \lim_{a\to 0} \prod_k \int \mathcal{D}u \exp\left\{ i \int dt \left[J_k(t)u(t)\sqrt{a} + \frac{1}{2}\dot{u}^2(t) - \frac{1}{2}(m^2 - i\varepsilon)\,u^2(t) - g_0 u^4(t) \right] \right\}$$

$$= \lim_{a\to 0} \prod_k < \exp\{i \int dt\, J_k(t)u(t)\sqrt{a}\} >$$

$$= \lim_{a\to 0} \prod_k \{ 1 - \frac{a}{2} < [\int J_k(t)u(t)\, dt\,]^2 > + \frac{a^2}{4!} < [\int J_k(t)u(t)\, dt\,]^4 > + \cdots \}$$

$$= \exp\{-1/2 \int d^3x\, dt\, dt'\, J(x,t)\, J(x,t') < u(t)\, u(t') > \}. \qquad (5.3)$$

Evidently (5.3) is a Gaussian result in which (assuming $0 < \tilde{g} \equiv \lim_{a\to 0} g_0(a) < \infty$)

$$< u(t)\, u(t') > = n \int \mathcal{D}u\, u(t)\, u(t') \exp\left\{ i \int dt \left[\frac{1}{2}\dot{u}^2(t) - \frac{1}{2}(m^2 - i\varepsilon)\, u^2(t) - \tilde{g}u^4(t) \right] \right\}$$

$$= < 0 \mid Q \exp(-i \mid t-t' \mid \tilde{H})\, Q \mid 0 >$$

$$= \sum_n < 0 \mid Q \mid n > e^{-i \mid t-t' \mid m_n} < n \mid Q \mid 0 >$$

$$= \sum_n \frac{\rho_n}{2m_n} e^{-i \mid t-t' \mid m_n}, \qquad (5.4)$$

where m_n, $n=1, 2, 3, \cdots$ denote the eigenvalues of $\tilde{H}$, $\frac{\rho_n}{2m_n} \equiv \mid < 0 \mid Q \mid n > \mid^2$. $\tilde{H} = 1/2\,(P^2 + m^2Q^2\,) + \tilde{g}Q^4$ - const. , and the constant is chosen so that $\tilde{H} \mid 0 > = 0$. It is easy to verify that $\sum_n \rho_n = 1$, and since $< 0 \mid Q\tilde{H}^4 Q \mid 0 > < \infty$ it follows that $\sum_n^{\infty} \rho_n m_n^3 < \infty$. Thus we have

$$Z[J] = \exp\{-1/4 \int d^3x\, dt\, dt'\, J(x,t)\, J(x,t') \sum_n (\rho_n/m_n)\, e^{-i \mid t-t' \mid m_n}\}$$

$$= \prod_n \exp\{-\frac{\rho_n}{4m_n}\int d^3x\, dt\, dt'\, J(x,t)\, J(x,t')\, e^{-i|t-t'|m_n}\}. \qquad (5.5)$$

The ultralocal free field corresponds to $g=\tilde{g}=0$, in which case $m_n = mn$, $\rho_1 = 1$, $\rho_n = 0$ $(n\neq 1)$ in (5.4), so that

$$< u(t)\, u(t') > = \frac{1}{2m} e^{-i|t-t'|m}.$$

Then (5.3) gives

$$Z_F[J] = \exp\{-\frac{1}{4m}\int d^3x\, dt\, dt'\, J(x,t)\, J(x,t')\, e^{-i|t-t'|m}\}$$

$$= <0 \mid T\exp\{i\int d^4x\, J(x,t)\Phi_F^m(x,t)\} \mid 0>. \qquad (5.6)$$

Observe that the ultralocal free field operator $\Phi_F^m(x,t)$ satisfies $\ddot{\Phi}_F^m(x,t) + m^2\Phi_F^m(x,t) = 0$.

Comparing (5.5) and Eq. (5.6), we get

$$Z[J] = \prod_n <0 \mid T\exp\{i\int d^4x\, \sqrt{\rho_n}\, J(x,t)\Phi_F^{m_n}(x,t)\} \mid 0>$$

$$= <0 \mid T\exp\{i\int d^4x\, J(x,t)\sum_n \sqrt{\rho_n}\Phi_F^{m_n}(x,t)\} \mid 0>$$

$$\equiv < 0 \mid T\exp\{ i\int d^4x\ J(x,t)\,\Phi(x,t)\} \mid 0 >, \qquad (5.7)$$

where $\Phi(x,t) = \sum_n \sqrt{\rho_n}\Phi_F^{m_n}(x,t)$. $\Phi_F^{m_n}(x,t)$ are ultralocal free field operators with mass m_n , $n = 1, 2, 3\cdots$,which satisfy $\ddot{\Phi}_F^{m_n}(x,t) + m_n^2\,\Phi_F^{m_n}(x,t) = 0$ for each n. From (5.7) we see that our regularized interacting model has led in the continuum limit to an ultralocal generalized free field, a natural analog of the relativistic generalized free field[17]. As a consequence of $[\dot{\Phi}(x,t), [\ddot{\Phi}(y,t), \dot{\Phi}(z,t)]\,] = 0$, which holds for a generalized free field under our present conditions, it is clear that the field operator Φ of an interaction like $V_I(\Phi) = g\Phi^4$ with g>0 can not satisfy this commutation equation. Obviously this (generalized) free-field behavior is limited neither to $g\Phi^4$ (which could be replaced by other local powers), or to the space-time dimension n=4. This fact indicates some sort of failure of the conventional formulation of canonical field quantization for these models. Moreover, with the help of coherent state techniques[18,19] we find that the classical limit of an ultralocal generalized free field does not limit as $(\hbar \to 0)$ to a classical field that satisfies the classical nonlinear canonical Eq.(1.2); instead, the classical limit of $\Phi(x,t) = \sum_n \sqrt{\rho_n}\Phi_F^{m_n}(x,t)$is given, not surprisingly, by $\varphi(x,t) = \sum_n \sqrt{\rho_n}\varphi^{m_n}(x,t)$, where $\ddot{\varphi}^{m_n}(x,t) + m_n^2\,\varphi^{m_n}(x,t) = 0$ for each n.

CHAPTER 6
OPERATOR ANALYSIS OF MULTI-COMPONENT ULTRALOCAL MODELS

The quantum theory of ultralocal scalar fields had been discussed extensively by Klauder[9,10] over two decades ago. The employment of probability theory and Hilbert space methods with an emphasis on infinitely divisible distributions and coherent states techniques, respectively, enables us to give a proper quantization of these models, which otherwise are meaningless within the conventional canonical formulation of quantum field theory (see chap. 5).

An operator solution of multi-component, nonrenormalizable, ultralocal quantum field models is developed here along lines presented earlier for single-component models. In §6.2, we will show that the additional, nonclassical, repulsive potential that is always present in the solution of the single-component case becomes indefinite and may even vanish in the multi-component case. The disappearance of that nonclassical and singular potential does not mean a return to standard field theory. The operator solution of multi-component ultralocal fields remains noncanonical. In §6.3, we will show that nontrivial, i.e., nongaussian, results hold for any number N of components, and suitable nontrivial behavior persists even in the infinite-component ($N=\infty$) case as well.

6.1 Operator Analysis of Single-Component Ultralocal Models

Let us briefly summarize the operator analysis of single-component ultralocal models[9,10]. Assume that the field operator $\Phi(x)$ becomes self-adjoint after smearing with a real test function $f(x)$ at sharp time. Since distinct spatial points characterize independent fields for all time, we may write the expectation functional as

$$E[f] \equiv < 0 \mid \exp\{ i \int dx\, \Phi(x)\, f(x) \} \mid 0 > = e^{-\int dx\, L[f(x)]}$$

$$= \exp\{ - \int dx \int d\lambda\, [\, 1 - e^{i\lambda f(x)}]\, c^2(\lambda) \}. \qquad (6.1)$$

Where in the last step, we have used the result of Levy's canonical representation theorem for the infinitely divisible characteristic functions[13]. and have eliminated a possible contribution of the Gaussian component, which applies to the free field. The real, even function $c(\lambda)$ is called the "model function".

An operator realization for the field $\Phi(x)$ is straightforward. Let $A^{\dagger}(x, \lambda)$ and $A(x, \lambda)$ denote conventional, irreducible Fock representation operators for which $|0>$ is the unique vacuum, $A(x, \lambda)\,|0> = 0$, for all $x \in \Re^{n-1}$, $\lambda \in \Re$. The only nonvanishing commutator is given by

$$\left[A(x, \lambda)\,,\, A^{\dagger}(x', \lambda')\right] = \delta(x-x')\, \delta(\lambda-\lambda'). \qquad (6.2)$$

Introduce the translated Fock operators

$$B(x,\lambda) = A(x,\lambda) + c(\lambda), \qquad B^{\dagger}(x,\lambda) = A^{\dagger}(x,\lambda) + c(\lambda), \tag{6.3}$$

Obviously, the operators $B^{\dagger}(x,\lambda)$ and $B(x,\lambda)$ follow the same commutation relation (6.2). Then the operator realization for the field $\Phi(x)$ is given by

$$\Phi(x) = \int d\lambda\, B^{\dagger}(x,\lambda)\, \lambda\, B(x,\lambda). \tag{6.4}$$

The correctness of this expression relies on the fact that

$$< 0 \mid \exp\{ i \int dx \Phi(x) f(x) \} \mid 0 >$$

$$= < 0 \mid \exp\{ i \int dx \int d\lambda\, B^{\dagger}(x,\lambda)\, \lambda f(x) B(x,\lambda) \} \mid 0 >$$

$$= < 0 \mid : \exp\{ \int dx \int d\lambda\, B^{\dagger}(x,\lambda)\, (e^{i\lambda f(x)} - 1) B(x,\lambda) \} : \mid 0 >$$

$$= \exp\{ - \int dx \int d\lambda\, [\, 1 - e^{i\lambda f(x)}]\, c^2(\lambda) \}$$

as required by (6.1).

The Hamiltonian operator is constructed from the creation and annihilation operators and is given by

$$H = \int dx \int d\lambda\; B^{\dagger}(x,\lambda)\, h(\partial/\partial\lambda, \lambda)\, B(x,\lambda)$$

$$= \int dx \int d\lambda\, A^{+}(x, \lambda)\, h(\partial/\partial\lambda, \lambda)\, A(x, \lambda). \qquad (6.5)$$

Where $h(\partial/\partial\lambda, \lambda) = -\frac{1}{2}\hbar^2 \partial^2/\partial\lambda^2 + V(\lambda)$, which is a self-adjoint operator in the λ variable alone. It is necessary that $h(\partial/\partial\lambda, \lambda) > 0$ in order that $H \geq 0$ and that $|0>$ be a unique ground state. Equality of the two expressions in (6.5) requires $h(\partial/\partial\lambda, \lambda)\, c(\lambda) = 0$, implying that $c(\lambda) \notin L^2$ in order for $|0>$ to be unique. This relation also determines $V(\lambda)$ as $V(\lambda) = \frac{1}{2}\hbar^2\, c''(\lambda)/c(\lambda)$. Assuming that $\int d\lambda\, \lambda^2\, c^2(\lambda) < \infty$, so that $< 0 \,|\Phi^2(f)\,| 0 > < \infty$, where $\Phi(f) = \int dx\, \Phi(x)\, f(x)$, together with the condition $\int d\lambda\, c^2(\lambda) = \infty$, we are led to choose

$$c(\lambda) = \frac{1}{|\lambda|^{\gamma}} \exp\left\{ -\frac{1}{2\hbar} m\lambda^2 - y(\lambda) \right\}, \qquad (6.6)$$

where γ, called the "singularity parameter", satisfies $1/2 \leq \gamma < 3/2$. Thus we have $(p_\lambda \equiv -i\hbar\, \partial/\partial\lambda)$

$$h(\partial/\partial\lambda, \lambda) = -\frac{1}{2}\hbar^2 \frac{\partial^2}{\partial\lambda^2} + \frac{\gamma(\gamma+1)\hbar^2}{2\lambda^2} + m(\gamma - 1/2)\hbar + \frac{1}{2}m^2\lambda^2 + V_I(\lambda)$$

$$= \frac{1}{2} p_\lambda^2 + \frac{\gamma(\gamma+1)\hbar^2}{2\lambda^2} + m(\gamma - 1/2)\hbar + \frac{1}{2}m^2\lambda^2 + V_I(\lambda), \qquad (6.7)$$

$$V_I(\lambda) = \frac{\hbar^2}{2}(-y'' + y'^2) + (m\hbar\lambda + \gamma\hbar^2/\lambda)\, y'. \qquad (6.7b)$$

This equation determines $c(\lambda)$ for any given interaction potential $V_I(\lambda)$, at least in principle. Notice that in addition to the free term $\frac{1}{2}p_\lambda^2 + \frac{1}{2}m^2\lambda^2$ and

the interaction term $V_I(\lambda)$, there appears a nonvanishing, positive, singular and nonclassical potential $\gamma(\gamma+1)\hbar^2/2\lambda^2$. It is important to note that this additional potential makes the path-integral formulation of ultralocal fields totally different than that of standard quantum field theory[14].

The definition of renormalized local powers of the field follows from the operator product expansion

$$\Phi(x)\,\Phi(y) = \delta(x-y)\int d\lambda\, B^{\dagger}(x,\lambda)\,\lambda^2\, B(x,\lambda) + :\Phi(x)\,\Phi(y):\ ,$$

which suggests the definition

$$\Phi_r^2(x) \equiv \int d\lambda\, B^{\dagger}(x,\lambda)\,\lambda^2\, B(x,\lambda) \equiv Z\,\Phi^2(x)\ , \quad Z^{-1} = \delta(0) \tag{6.8}$$

or more generally,

$$\Phi_r^k(x) \equiv \int d\lambda\, B^{\dagger}(x,\lambda)\,\lambda^k\, B(x,\lambda) \equiv \frac{1}{Z}[Z\,\Phi(x)]^k,\ \ k = 1, 2, 3, \cdots \tag{6.9}$$

For $k \geq 1$, these expressions are local operators, i.e., become operators when smeared by a test function. For $k \leq 0$, let $\Phi^{-1}(x) \equiv Z^2\int d\lambda\, B^{\dagger}(x,\lambda)\,\frac{1}{\lambda}\,B(x,\lambda)$, and $\int d\lambda\, B^{\dagger}(x,\lambda)\,\frac{1}{\lambda}\,B(x,\lambda) \equiv \Phi_r^{-1}(x)$, then we can extend (6.9) for renormalized negative powers of the field. But we should note that for $k \leq 0$, $\Phi_r^k(x)$ are not local operators.

Corresponding to (6.7), the Hamiltonian may be expressed by renormalized fields as

$$H = \int \left\{ \frac{1}{2}\Pi_r^2(x) + \frac{1}{2}m^2\Phi_r^2(x) + V_I[\Phi_r(x)] \right\} dx, \tag{6.10}$$

where $\Pi_r^2 = \dot{\Phi}_r^2 + \gamma(\gamma+1)\hbar^2 \Phi_r^{-2} + m(2\gamma - 1)\hbar \Phi_r^0$. Obviously it means that Π_r neither fits into the canonical framework($\Pi_r = \dot{\Phi}_r$) nor fulfills the standard canonical commutation relation $[\Phi_r(x), \Pi_r(y)] = i\hbar\delta(x - y)$. We should also notice that although $\dot{\Phi}_r$ ($= \dot{\Phi} = -i[\Phi, H]/\hbar = -i\hbar \int d\lambda\ B^\dagger(x, \lambda)\, \partial/\partial\lambda\, B(x, \lambda)$) is not a local operator due to the assumption regarding $c(\lambda)$, neither are $\dot{\Phi}_r^2, \Phi_r^{-2}, \Phi_r^0$ local operators by themselves alone; only the combination Π_r^2 is a well-defined local operator. These are major differences from standard quantum field theory.

The Heisenberg field operator is given by

$$\Phi(x,t) = e^{iHt/\hbar}\, \Phi(x)\, e^{-iHt/\hbar}$$

$$= \int d\lambda\, B^\dagger(x, \lambda)\, e^{iht/\hbar\lambda} e^{-iht/\hbar}\, B(x, \lambda)$$

$$= \int d\lambda\, B^\dagger(x, \lambda)\, \lambda(t)\, B(x, \lambda) \tag{6.11}$$

which is well defined. The time-ordered truncated n-point vacuum expectation values are given by

$$< 0 | T\, [\Phi(x_1,t_1)\, \Phi(x_2,t_2) \cdots \Phi(x_n,t_n)] | 0 >^T$$

$$= \delta(x_1-x_2)\,\delta(x_2-x_3)\cdots\delta(x_{n-1}-x_n)\cdot\int d\lambda\; c(\lambda)T\,[\,\lambda(t_1)\,\lambda(t_2)\cdots\lambda(t_n)]\,c(\lambda). \quad (6.12)$$

For ultralocal pseudo-free fields (defined when the interaction potential $V_I(\lambda)$ vanishes), we have the model function $c(\lambda) = |\lambda|^{-\gamma}\exp\left\{-m\lambda^2/2\hbar\right\}$, and the expectation functional

$$E[f] = \exp\{-\int dx\int d\lambda\;\frac{1-\cos\,[\,\lambda f(x)\,]}{|\lambda|^{2\gamma}}\exp\left\{-\frac{1}{\hbar}m\lambda^2\right\}, \quad (6.13)$$

which is obviously not a Gaussian solution such as for ultralocal free fields.

In §6.2, the operator analysis of finite-component ultralocal models is presented. There we will see how an indefinite (positive, negative or vanishing), singular, and nonclassical potential affects the operator solutions.

In §6.3, the expectation functional of infinite-component ultralocal fields is discussed. Even in this case the solution of any interacting theory does not reduce to that of the free theory in the limit of vanishing nonlinear interaction and again supports the concept of a pseudo-free theory[7,8].

6.2 Operator Analysis of Finite-Component Ultralocal Fields

Following the pattern of the single-component case, the field operator and the Hamiltonian of N-component ultralocal fields are defined by

$$\vec{\Phi}(x) = \int d\vec{\lambda}\, B^{\dagger}(x, \vec{\lambda})\, \vec{\lambda}\, B(x, \vec{\lambda}) \tag{6.14}$$

and

$$\begin{aligned} H &= \int dx \int d\vec{\lambda}\; B^{\dagger}(x, \vec{\lambda})\, h(\vec{\nabla}_{\lambda}, \vec{\lambda})\, B(x, \vec{\lambda}) \\ &= \int dx \int d\vec{\lambda}\, A^{\dagger}(x, \vec{\lambda})\, h(\vec{\nabla}_{\lambda}, \vec{\lambda})\, A(x, \vec{\lambda}) \end{aligned} \tag{6.15}$$

where $h(\vec{\nabla}_{\lambda}, \vec{\lambda}) = -\frac{1}{2}\hbar^2\nabla_{\lambda}^2 + V(\lambda)$, $\lambda \equiv \sqrt{\vec{\lambda}^2}$. Notice that here the N-component ultralocal models under consideration have $O(N)$ symmetry. The commutation relation becomes

$$\left[A(x, \vec{\lambda})\,,\, A^{\dagger}(x', \vec{\lambda}')\right] = \delta(x-x')\, \delta(\vec{\lambda}-\vec{\lambda}') \tag{6.16}$$

and the unique ground state satisfies $A(x, \vec{\lambda})|0> = 0$. The operator $B(x, \vec{\lambda})$ is related to $A(x, \vec{\lambda})$ by

$$B(x, \vec{\lambda}) = A(x, \vec{\lambda}) + c(\lambda). \tag{6.17}$$

The renormalized local powers of the field are given by

$$\Phi^{k}_{r\,i_1\cdots i_k}(x) \equiv \int d\vec{\lambda}\, B^{\dagger}(x,\vec{\lambda})\,\lambda_{i_1}\cdots\lambda_{i_k}\,B(x,\vec{\lambda}), \quad i_1,\cdots,i_k = 1,2,3,\cdots N$$

Occasionally, we need to use the renormalized negative powers of the field even though they are not local operators. Take a single-component case for an example:

$$\Phi^{k}_{ri}(x) \equiv \int d\vec{\lambda} B^{\dagger}(x,\vec{\lambda})\lambda_i^{k} B(x,\vec{\lambda}) \equiv \frac{1}{Z}[Z\,\Phi_i(x)]^{k}, \quad i = 1,2\cdots N, \tag{6.18}$$

which can be extended to $k \le 0$, as long as we define $\Phi_i^{-1}(x) \equiv Z^2 \int d\vec{\lambda} B^{\dagger}(x,\vec{\lambda})\frac{1}{\lambda_i} B(x,\vec{\lambda})$ and $\int d\vec{\lambda} B^{\dagger}(x,\vec{\lambda})\frac{1}{\lambda_i} B(x,\vec{\lambda}) \equiv \Phi^{-1}_{ri}(x)$.

From the requirement that $h(\vec{\nabla}_\lambda, \vec{\lambda})\, c(\lambda) = 0$, we have $V(\lambda) = \frac{1}{2}\hbar^2 \nabla^2_\lambda c(\lambda)/c(\lambda)$. The real, even model function $c(\lambda)$ becomes

$$c(\lambda) = \frac{1}{\lambda^{\Gamma}} \exp\left\{ -\frac{1}{2\hbar} m\lambda^2 - y(\lambda) \right\} \tag{6.19}$$

with $N/2 \le \Gamma < N/2 + 1$ which is determined by the conditions $\int d^N\lambda\; c^2(\lambda) = \infty$ and $\int d^N\lambda\, \lambda^2\, c^2(\lambda) < \infty$.

Next let us find out how the operator $h(\vec{\nabla}_\lambda, \vec{\lambda})$ appears. From (6.19), it follows that

$$V(\lambda) = \frac{1}{2}\hbar^2 \frac{\nabla^2_\lambda c(\lambda)}{c(\lambda)} = \frac{\hbar^2}{2\,c(\lambda)} \sum_{i=1}^{N} \frac{\partial^2}{\partial \lambda_i^2} c(\lambda)$$

$$= \frac{\Gamma(\Gamma+2-N)\hbar^2}{2\lambda^2} + m\hbar\,(\Gamma - N/2) + \frac{1}{2}m^2\lambda^2 + V_I(\lambda)\,, \qquad (6.20)$$

where $V_I(\lambda) = \frac{1}{2}\hbar^2\,(-y'' + y'^2) + y'\,\{m\hbar\lambda + [\Gamma - (N-1)/2]\,\hbar^2/\lambda\}$, which corresponds to the interaction. Therefore

$$h(\vec{\nabla}_\lambda, \vec{\lambda}) = -\frac{1}{2}\hbar^2\nabla_\lambda^2 + V(\lambda)$$

$$= -\frac{1}{2}\hbar^2\nabla_\lambda^2 + \frac{\Gamma(\Gamma+2-N)\hbar^2}{2\lambda^2} + m\hbar\,(\Gamma - N/2) + \frac{1}{2}m^2\lambda^2 + V_I(\lambda). \qquad (6.21)$$

This base Hamiltonian of multi-component ultralocal fields is similar to that of single-component ultralocal fields (6.7); but a surprising difference between the two equations is that unlike the non-vanishing, positive, singular and nonclassical potential $\gamma(\gamma+1)\hbar^2/2\lambda^2$ appearing in (6.7), $\Gamma(\Gamma+2-N)\hbar^2/2\lambda^2$ is not always positive and moreover it may vanish. These properties follow since $N/2 \le \Gamma < N/2 + 1$, i.e., $0 \le \Gamma - N/2 < 1$ and therefore

$$2 - N/2 \le \Gamma+2 - N = 2 + (\Gamma - N/2) - N/2 < 3 - N/2.$$

Summarizing, $\Gamma+2-N>0$ for $N<4$; $\Gamma+2-N<0$ for $N\ge6$; and $\Gamma+2-N$ is indefinite, when $4\le N<6$, i.e., it may be larger than zero, smaller than zero, or even equal to zero. As examples of vanishing $\Gamma(\Gamma+2-N)\hbar^2/2\lambda^2$ we have $N=4$ and $\Gamma=2$, or $N=5$ and $\Gamma=3$.

It is worth mentioning that the disappearance of $\Gamma(\Gamma+2-N)\hbar^2/2\lambda^2$ does not mean a return to standard (canonical) field theory or standard path-

integral formulation; even in this case it can be shown that a similar non-vanishing, singular, nonclassical potential arises in the regularized path-integral formulation (Chapter 8).

It is worthwhile to discuss further the Hamiltonian which according to (6.21), is given by

$$H = \int \left\{ \frac{1}{2}\vec{\Pi}_r^{\rightarrow 2}(x) + \frac{1}{2}m^2\vec{\Phi}_r^{\rightarrow 2}(x) + V_I[\vec{\Phi}_r(x)] \right\} dx. \qquad (6.22)$$

Here $\vec{\Pi}_r^{\rightarrow 2}(x) = \dot{\vec{\Phi}}_r^{\dot{\rightarrow} 2}(x) + \Gamma(\Gamma+2-N)\hbar^2 |\vec{\Phi}_r(x)|^{-2} + m\hbar(2\Gamma - N)|\vec{\Phi}_r(x)|^0$. Notice that $\dot{\Phi}_{ri}^2(x)\left[= \int d\vec{\lambda}\, B^\dagger(x,\vec{\lambda})(-\hbar^2\partial^2/\partial\lambda_i^2)B(x,\vec{\lambda}) \right]$, $i=1,2,\cdots N$, are not local operators, neither are $\dot{\Phi}_{ri}(x)$, $|\vec{\Phi}_r(x)|^0$ or $|\vec{\Phi}_r(x)|^{-2}$, but somewhat surprisingly $\sum_{i=1}^{N} \dot{\Phi}_{ri}^2(x) \equiv \dot{\vec{\Phi}}_r^{\dot{\rightarrow} 2}(x)$ is a well-defined operator for $N=4$ and $\Gamma=2$ and $\sum_{i=1}^{N} \dot{\Phi}_{ri}^2(x) + m\hbar\,|\vec{\Phi}_r|^0 \equiv \dot{\vec{\Phi}}_r^{\dot{\rightarrow} 2}(x) + m\hbar\,|\vec{\Phi}_r|^0$ for $N=5$ and $\Gamma=3$. In the case of $N=4$ and $\Gamma=2$, we have $\vec{\Pi}_r^{\rightarrow 2}(x) = \dot{\vec{\Phi}}_r^{\dot{\rightarrow} 2}(x)$, which in the present case is a local operator, but $\vec{\Pi}_r(x) = \dot{\vec{\Phi}}_r(x)$ involves local forms rather than local operators due to the ill defined $\dot{\Phi}_{ri}(x)$.

Summarizing, the canonical equation $\vec{\Pi}_r(x) = \dot{\vec{\Phi}}_r(x)$ and the standard canonical commutation relation $[\Phi_{ri}(x), \Pi_{rj}(y)] = i\hbar\delta_{ij}\delta(x-y)$ do not hold for the multi-component ultralocal fields, just as for the single-component ultralocal fields, irrespect of whether the singular potential $\Gamma(\Gamma+2-N)\hbar^2/2\lambda^2$ $[\sim |\vec{\Phi}_r(x)|^{-2}]$ is present or not.

Analogous to the single-component case, the Heisenberg field operator is given by

$$\vec{\Phi}(x,t) = e^{iHt/\hbar}\,\vec{\Phi}(x)\,e^{-iHt/\hbar}$$

$$= \int d\vec{\lambda}\; B^{\dagger}(x,\vec{\lambda})\, e^{iht/\hbar}\vec{\lambda} e^{-iht/\hbar}\, B(x,\vec{\lambda})$$

$$= \int d\vec{\lambda}\; B^{\dagger}(x,\vec{\lambda})\,\vec{\lambda}(t)\, B(x,\vec{\lambda}) \qquad (6.23)$$

which is well defined. It follows that the time-ordered, truncated n-point function reads

$$< 0 \mid T\,[\Phi_{i_1}(x_1,t_1)\,\Phi_{i_2}(x_2,t_2)\cdots\Phi_{i_n}(x_n,t_n)] \mid 0 >^{T}$$

$$= \delta(x_1-x_2)\,\delta(x_2-x_3)\cdots\delta(x_{n-1}-x_n)\int d\vec{\lambda}\; c(\lambda)\; T[\lambda_{i_1}(t_1)\,\lambda_{i_2}(t_2)\cdots\lambda_{i_n}(t_n)]\; c(\lambda). \qquad (6.24)$$

6.3 Operator Analysis of Infinite-Component Ultralocal Fields

The expectation functional of finite-component fields is

$$E[\vec{f}] \equiv < 0 \mid \exp\left[i \int dx\, \vec{\Phi}(x) \cdot \vec{f}(x) \right] \mid 0 >$$

$$= \exp \{ - g \int dx \int d\vec{\lambda}\, [1 - e^{i\vec{\lambda} \cdot \vec{f}(x)}]\, c^2(\lambda) \}. \qquad (6.25)$$

Here we have introduced g, a scale factor which cannot be determined on general grounds but rather represents the only arbitrary renormalization scale involved in the operator construction. The possibility of its existence lies in the fact that two model functions differing by a constant factor lead to the same differential operator h and thus to the same Hamiltonian H.

An operator realization of the infinite-component field requires us to give a proper measure which is well defined. In doing so, notice that we can absorb the model function $c(\lambda)$ into the measure so that we have the new measure $d\rho(\vec{\lambda}) = g\, c^2(\lambda)\, \Pi d\lambda_i$. Let us redefine the commutator

$$\left[A(x, \vec{\lambda}) , A^{+}(x', \vec{\lambda}') \right] = \delta(x-x')\, \delta_\rho(\vec{\lambda}; \vec{\lambda}'), \qquad (6.26)$$

where $\delta_\rho(\vec{\lambda}; \vec{\lambda}')$ is related to the measure ρ by $\int d\rho(\vec{\lambda}')\, f(\vec{\lambda}')\delta_\rho(\vec{\lambda}; \vec{\lambda}') = f(\vec{\lambda})$. The operator B and the infinite-component field are redefined as

$$B(x, \vec{\lambda}) = A(x, \vec{\lambda}) + 1, \tag{6.27}$$

$$\vec{\Phi}(x) = \int d\rho(\lambda)\, B^{\dagger}(x, \vec{\lambda})\, \vec{\lambda}\, B(x, \vec{\lambda}). \tag{6.28}$$

Then we immediately recognize that (6.25) stays the same. The renormalized local powers of the field and the Hamiltonian are given analogously by

$$\Phi^{k}_{r\, i_1 \cdots i_k}(x) \equiv \int d\rho(\lambda)\, B^{\dagger}(x, \vec{\lambda})\, \lambda_{i_1} \cdots \lambda_{i_k}\, B(x, \vec{\lambda}), \quad i_1, \cdots, i_k = 1, 2, 3, \cdots N, \tag{6.29}$$

and

$$H = \int dx \int d\rho(\lambda)\, B^{\dagger}(x, \vec{\lambda})\, h(\overrightarrow{\nabla_{\lambda}}, \vec{\lambda})\, B(x, \vec{\lambda})$$

$$= \int dx \int d\rho(\lambda)\, A^{\dagger}(x, \vec{\lambda})\, h(\overrightarrow{\nabla_{\lambda}}, \vec{\lambda})\, A(x, \vec{\lambda}), \tag{6.30}$$

with the requirement of $h(\overrightarrow{\nabla_{\lambda}}, \vec{\lambda}) \cdot 1 = 0$. The Heisenberg field operator is given by

$$\vec{\Phi}(x,t) = e^{iHt/\hbar}\, \vec{\Phi}(x)\, e^{-iHt/\hbar}$$

$$= \int d\rho(\vec{\lambda})\, B^{\dagger}(x, \vec{\lambda})\, e^{iht/\hbar} \vec{\lambda} e^{-iht/\hbar}\, B(x, \vec{\lambda})$$

$$= \int d\rho(\lambda)\, B^{\dagger}(x, \vec{\lambda})\, \vec{\lambda}(t)\, B(x, \vec{\lambda}). \tag{6.31}$$

The time-ordered, truncated n-point function reads

$$< 0 \mid T\,[\Phi_{i_1}(x_1,t_1)\,\Phi_{i_2}(x_2,t_2)\cdots\Phi_{i_n}(x_n,t_n)] \mid 0 >^{T}$$

$$= \delta(x_1-x_2)\,\delta(x_2-x_3)\cdots\delta(x_{n-1}-x_n)\int d\rho(\lambda)\,T[\lambda_{i_1}(t_1)\,\lambda_{i_2}(t_2)\cdots\lambda_{i_n}(t_n)]\cdot 1. \qquad (6.32)$$

It is worth mentioning that the above formulation for the infinite-component case is suitable for the finite component case as well.

Next we try to obtain the characteristic functional of infinite-component fields as the limit of finite-component ones. Take the general form of $c(\lambda) = \lambda^{-\Gamma}\exp\left\{-\frac{1}{2\hbar}m\lambda^2 - y(\lambda)\right\}$ in the finite case, then we have

$$E[\vec{f}] = \exp\left\{-g\int dx\int d\vec{\lambda}\,[1-e^{i\vec{\lambda}\cdot\vec{f}(x)}]\frac{1}{\lambda^{2\Gamma}}\exp\left[-\frac{1}{\hbar}m\lambda^2 - 2y(\lambda)\right]\right\} \qquad (6.33)$$

In order for this limit to exist as $N\to\infty$, we need to scale several parameters, namely $y=y_N$, $m=m_N$ and to choose a suitable factor $g=g_N$. Following the Ref. 22, suppressing the integration over x, let us consider

$$L[\vec{f}] \equiv lim_{N\to\infty}\, g_N\int d\vec{\lambda}\,[1-e^{i\vec{\lambda}\cdot\vec{f}}]\frac{1}{\lambda^{2\Gamma}}\exp\left\{-\frac{1}{\hbar}m_N\lambda^2 - 2y_N(\lambda)\right\} \qquad (6.34)$$

$$= lim_{N\to\infty}\frac{g_N}{(\Gamma-1)!}\int_0^{\infty} dr\, r^{\Gamma-1}\int d\vec{\lambda}\,[1-e^{i\vec{\lambda}\cdot\vec{f}}]\exp\left\{-(\frac{1}{\hbar}m_N + r)\,\lambda^2 - 2y_N(\lambda)\right\}.$$

where $\int_0^\infty dr\, r^n e^{-\gamma r} = \frac{n!}{\gamma^{n+1}}$ has been used. If we introduce the Fourier transform pair: $e^{-2y_N(\lambda)} = \int d\xi\, h_N(\xi)\, e^{i\xi\lambda^2}$, $h_N(\xi) = \frac{1}{2\pi}\int d\lambda^2\, e^{-2y_N(\lambda)}\, e^{-i\xi\lambda^2}$, $L[\vec{f}]$ can be expressed as

$$lim_{N\to\infty} \frac{g_N}{(\Gamma-1)!}\int_0^\infty dr\, r^{\Gamma-1}\int d\vec{\lambda}\,[1 - e^{i\vec{\lambda}\cdot\vec{f}}]\exp\left\{-(\frac{1}{\hbar}m_N + r)\,\lambda^2 + i\xi\lambda^2\right\}h_N(\xi)\,d\xi$$

The integration over $\vec{\lambda}$ now involves simply Gaussian integration $\left(\int d\vec{x}\, e^{-\alpha\vec{x}^2 + i\vec{f}\cdot\vec{x}} = \left(\frac{\pi}{\alpha}\right)^{N/2} e^{-\vec{f}^2/4\alpha}\right)$ and leads to

$$lim_{N\to\infty} \frac{g_N}{(\Gamma-1)!}\int_0^\infty dr\, r^{\Gamma-1}\left(\frac{\pi}{m_N/\hbar + r - i\xi}\right)^{N/2}[1 - e^{-\vec{f}^2/\,4(m_N/\hbar + r - i\xi)}]\,h_N(\xi)\,d\xi,$$

$$= lim_{N\to\infty} \frac{\pi^{N/2} g_N}{(\Gamma-1)!}\int_0^\infty dr\, r^{\theta}(m_N/r\hbar - i\xi/r + 1)^{-N/2}[1 - e^{-\vec{f}^2/\,4(m_N/\hbar + r - i\xi)}]\,h_N(\xi)\,d\xi,$$

where $-1 \le \theta \equiv \Gamma - 1 - N/2 < 0$. Now we are ready to make the choice of y_N, m_N and g_N. Assume that $g_N = g\pi^{-N/2}(\Gamma-1)!$, $m_N = m/N$ and $y_N(\lambda) = y(\lambda/\sqrt{N})$ which suggests that $h_N(\xi) = N\,h(N\xi)$. Following a change of variables of ξ, we take the limit $N\to\infty$, by using $lim_{N\to\infty}(1 + \frac{m/r\hbar - i\xi/r}{N})^{-N/2} = \exp\left\{-\frac{1}{2r}(m/\hbar - i\xi)\right\}$, and we get

$$L[\vec{f}] = g\int_0^\infty dr\, r^{\theta}\exp\left\{-\frac{1}{2r}(m/\hbar - i\xi)\right\}[1 - e^{-\vec{f}^2/\,4r}]\,h(\xi)\,d\xi$$

$$= g\int_0^\infty dr\, r^\theta \exp\left\{-\frac{1}{2r}(m/\hbar) - 2y(\frac{1}{2r})\right\}[1 - e^{-\vec{f}^2/4r}]$$

$$= g\int_0^\infty d\sigma \frac{2}{(2\sigma)^{\theta+2}} \exp\{-(m/\hbar)\sigma - 2y(\sigma)\}[1 - e^{-\sigma\vec{f}^2/2}], \qquad (6.35)$$

where $-1 \le \theta < 0$, and in the last step, we have substituted $\sigma = 1/(2r)$.

We can also try in the following way to obtain the characteristic functional of infinite-component fields as the limit of finite-component ones. Take the general form of $c(\lambda) = \lambda^{-\Gamma}\exp\left\{-\frac{1}{2\hbar}m\lambda^2 - y(\lambda)\right\}$ in the finite case, then we have

$$E[\vec{f}] = \exp\{-\int dx \int d\rho(\vec{\lambda})[1 - e^{i\vec{\lambda}\cdot\vec{f}(x)}]\}$$

$$= \exp\left\{-g\int dx \int d\vec{\lambda}[1 - e^{i\vec{\lambda}\cdot\vec{f}(x)}]\frac{1}{\lambda^{2\Gamma}}\exp[-\frac{1}{\hbar}m\lambda^2 - 2y(\lambda)]\right\} \qquad (6.36)$$

In order for this limit to exist as $N\to\infty$, we need to scale several parameters, namely $y=y_N$, $m=m_N$ and to choose a suitable factor $g=g_N$. Suppressing the integration over x, let us consider

$$L[\vec{f}] \equiv \int d\rho_N(\vec{\lambda})[1 - e^{i\vec{\lambda}\cdot\vec{f}(x)}] \qquad (6.37)$$

$$= lim_{N\to\infty}\, g_N \int d\vec{\lambda}[1 - e^{i\vec{\lambda}\cdot\vec{f}}]\frac{1}{\lambda^{2\Gamma}}\exp\left\{-\frac{1}{\hbar}m_N\lambda^2 - 2y_N(\lambda)\right\}$$

$$= lim_{N\to\infty}\, g_N \int d\lambda \frac{\lambda^{N-1}}{\lambda^{2\Gamma}} \exp\left\{ -\frac{1}{\hbar} m_N \lambda^2 - 2y_N(\lambda) \right\} \int d\Omega\, [\, 1 - e^{i\lambda f \cos\Theta}] \,. \qquad (6.38)$$

Where $d\Omega = \sin^{N-2}\Theta\, d\Theta\, d\Omega'$. By changing the variable $\Theta \to \tilde{\Theta}/N + \pi/2$ and taking N large, we have $\cos\Theta = -\sin(\tilde{\Theta}/N) \approx -\tilde{\Theta}/N$ and therefore $\sin^{N-2}\Theta = \cos^{N-2}(\tilde{\Theta}/N) \approx [\, 1 - \frac{1}{2}(\tilde{\Theta}/N)^2\,]^{N-2} \approx \exp[\, -\frac{1}{2}(\tilde{\Theta}/N)^2 (N-2)\,]$. Hence we may write

$$\int d\Omega\, [\, 1 - e^{i\lambda f \cos\Theta}] = \int d\Theta \exp[\, -\tfrac{1}{2}\Theta^2 (N-2)\,]\, [\, 1 - e^{-i\lambda f \Theta}] \int d\Omega'$$

$$= \left[\frac{2\pi}{N-2}\right]^{N/2} \int d\Omega' \cdot \{\, 1 - \exp[\, -\tfrac{1}{2} f^2 \lambda^2 / (N-2)]\,\}$$

$$= S \cdot \{\, 1 - \exp[\, -\tfrac{1}{2} f^2 \lambda^2 / (N-2)]\,\}. \qquad (6.39)$$

Where S denotes the spherical surface area in N-dimensions, $N >> 1$, namely $S = \left[\frac{2\pi}{N-2}\right]^{N/2} \int d\Omega'$. Substituting (6.39) into (6.38), we have

$$\vec{L[f]} = lim_{N\to\infty}\, g_N \int d\lambda \frac{1}{\lambda^{2\Gamma - N + 1}} \exp\left\{ -\frac{1}{\hbar} m_N \lambda^2 - 2y_N(\lambda) \right\} S \cdot \{1 - \exp[\, -\tfrac{1}{2} f^2 \lambda^2 / (N-2)]\,\}$$

$$= lim_{N\to\infty} g_N \int d\lambda \frac{N^{-(\mu-1)/2} S}{\lambda^{\mu}} \exp\left\{ -\frac{1}{\hbar} m_N N \lambda^2 - 2y_N(\sqrt{N}\,\lambda) \right\} \{1 - \exp[\, -\tfrac{1}{2} f^2 \lambda^2 \frac{N}{N-2}]\,\}.$$

In the last step we have made a change of variables of $\lambda \to \sqrt{N}\,\lambda'$, $\mu = 2\Gamma - N + 1$.

Now we must choose y_N, m_N and g_N. Assume that $m_N = m/N$, $y_N(\lambda) = y(\lambda/\sqrt{N})$ and $g_N = N^{(\mu-1)/2}\bar{g}/S$, where $\bar{g}$ is proportional to g in (6.35). By taking the limit $N \to \infty$, we obtain

$$L[\vec{f}] = \bar{g} \int d\lambda \frac{1}{\lambda^{\mu}} \exp\left\{-\frac{1}{\hbar} m\lambda^2 - 2y(\lambda)\right\} \cdot [1 - e^{-f^2\lambda^2/2}], \tag{6.40}$$

where $1 \leq \mu < 3$. So the characteristic functional of infinite-component fields becomes

$$E[\vec{f}] = \exp\{-\int dx \int d\rho(\vec{\lambda})\,[1 - e^{i\vec{\lambda}\cdot\vec{f}(x)}]\}$$

$$= \exp\left[-\bar{g}\int dx \int d\lambda \frac{1}{\lambda^{\mu}} \exp\left\{-\frac{1}{\hbar} m\lambda^2 - 2y(\lambda)\right\} \cdot [1 - e^{-f^2\lambda^2/2}]\right]. \tag{6.41}$$

From (6.41), we see that a nontrivial, i.e., non-Gaussian, result holds for infinite-component fields, just as it does for the finite-component ones.

Notice that the scalings we have used here, such as $m_N = m/N$ and $y_N(\lambda) = y(\lambda/\sqrt{N})$, are different from the standard ones. For example: for $V_I = \vec{\Phi}^4$, the standard scaling assumes $(V_I)_N = \vec{\Phi}^4/N$, while the non-standard scaling employs $m_N = m/N$ and $y_N(\lambda) = y(\lambda/\sqrt{N})$ which gives $(V_I)_N[\lambda] = V_I[\lambda/\sqrt{N}] = \lambda^4/N^2$ or equivalently $(V_I)_N[\Phi] = V_I[\Phi/\sqrt{N}] = \Phi^4/N^2$. The nonstandard scaling admits a *1/N* expansion relevant to the Poisson-distributed finite-*N* solution, while the standard scaling does not. Another similar non-standard scaling example was shown to hold in the independent-value models where $m_N^2 = m^2/N$, $(V_I)_N[\vec{\Phi}^2] = V_I[\vec{\Phi}^2/N]$ are also used[22].

Judging from these examples it would appear that the standard scaling may fail for certain nonrenormalized fields.

CHAPTER 7
THE CLASSICAL LIMIT OF ULTRALOCAL MODELS

As given in chapter 5, the classical Hamiltonian of an ultralocal scalar field may expressed as

$$H_{cl} = \int \left\{ \frac{1}{2}\pi^2(x) + \frac{1}{2}m^2\varphi^2(x) + V_I[\varphi(x)] \right\} d^3x \,, \tag{7.1}$$

where $x \in \Re^{n-1}$, π, φ denote classical fields, and $V_I[\varphi]$ ($= V_I[-\varphi]$, for simplicity) the interaction potential. The classical canonical equations of motion appropriate to (7.1) are

$$\dot{\varphi}(x,t) = \frac{\delta H_{cl}}{\delta \pi(x,t)} = \pi(x,t),$$

$$\dot{\pi}(x,t) = -\frac{\delta H_{cl}}{\delta \varphi(x,t)} = -m^2\varphi(x,t) - V_I'[\varphi(x,t)], \tag{7.2}$$

$$\ddot{\varphi}(x,t) = -m^2\varphi(x,t) - V_I'[\varphi(x,t)].$$

In fact, these formulas do not take into account a vestige of the quantum theory that really is part of the classical action. Since the additional, nonclassical, repulsive potential proportional to (field)$^{-2}$ in (6.10) belongs in

the quantum Hamiltonian density, as $\hbar \to 0$ the coefficient of this term vanishes save when $\varphi(x) = 0$. To account for this term we formally write $0 \cdot \varphi^{-2}(x,t)$ in the classical Hamiltonian density, and to respect this potential we need to derive the equations of motion for $\varphi(x,t)$ by means of a scale transformation[20], namely, using $\delta\varphi(x,t) = \delta S(x,t) \cdot \varphi(x,t)$. This leads to a related but alternative set of classical equations of motion (see Examples 1 and 2 below).

In the following, we will show that the alternative quantum theory of ultralocal scalar fields described in Chapter 6 does indeed lead to the required classical limit as $\hbar \to 0$, thereby giving additional support to such an alternative, non-Gaussian solution.

7.1 Ultralocal Fields and Associated Coherent States

The proper quantum Hamiltonian of ultralocal fields given by (6.10) has an evident connection with the classical Hamiltonian. In (6.10), the subscript 'r' means the fields are renormalized, and instead of $\Pi = \dot{\Phi}$, we have $\Pi_r^2 = \dot{\Phi}_r^2 + O(\hbar)\,\Phi_r^0 + O(\hbar^2)\,\Phi_r^{-2}$.

According to (6.4) and (6.5), the field operator Φ and Hamiltonian operator are given by

$$\Phi(x) = \int d\lambda\, B^{\dagger}(x,\lambda)\,\lambda\, B(x,\lambda)\,, \tag{7.3}$$

$$B(x,\lambda) = A(x,\lambda) + c(\lambda),$$

$$H = \int d^3x \int d\lambda\, A^{\dagger}(x, \lambda)\, h(\frac{\partial}{\partial\lambda}, \lambda)\, A(x, \lambda),$$

or

$$H = \int d^3x \int d\lambda\, B^{\dagger}(x, \lambda)\, h(\frac{\partial}{\partial\lambda}, \lambda)\, B(x, \lambda), \tag{7.4}$$

with $$h(\frac{\partial}{\partial\lambda}, \lambda) = -\frac{1}{2}\hbar^2\frac{\partial^2}{\partial\lambda^2} + \frac{\gamma(\gamma+1)\hbar^2}{2\lambda^2} + \frac{1}{2}m^2\lambda^2 + V_1(\lambda)$$

$$= \frac{1}{2}p_\lambda^2 + \frac{\gamma(\gamma+1)\hbar^2}{2\lambda^2} + \frac{1}{2}m^2\lambda^2 + V_1(\lambda)\,, \tag{7.5}$$

$$(\, p_\lambda \equiv -i\hbar\frac{\partial}{\partial\lambda}\,)$$

Here we write the interaction potential as $V_1(\lambda)$ instead of $V_I(\lambda)$, since we have used V_I as the classical interaction potential in (7.1). By using

$$\exp(\frac{i}{\hbar}Ht)\, B(x, \lambda) \exp(-\frac{i}{\hbar}Ht) = \exp[-\frac{i}{\hbar}t\, h(\frac{\partial}{\partial\lambda}, \lambda)\,]\, B(x, \lambda) \tag{7.6}$$

we can show that the Heisenberg field operator for any renormalized field power is given by

$$\Phi_r^{\theta}(x, t) = \int d\lambda\, B^{\dagger}(x, \lambda) \exp[\frac{i}{\hbar}t\, h(\frac{\partial}{\partial\lambda}, \lambda)\,]\, \lambda^{\theta} \exp[-\frac{i}{\hbar}t\, h(\frac{\partial}{\partial\lambda}, \lambda)\,]\, B(x, \lambda)$$

$$= \int d\lambda\, B^{\dagger}(x, \lambda)\, \lambda^{\theta}(t)\, B(x, \lambda)\,, \tag{7.7}$$

where $\lambda^{\theta}(t) = \exp[\frac{i}{\hbar} t\, h(\frac{\partial}{\partial\lambda}, \lambda)]\, \lambda^{\theta} \exp[-\frac{i}{\hbar} t\, h(\frac{\partial}{\partial\lambda}, \lambda)]$. Even though $[\Phi_r(x), \Pi_r(y)] \neq i\hbar\, \delta(x-y)$, the variables λ and p_λ satisfy the standard one-dimensional canonical commutation relations. Therefore

$$\dot{\lambda} = \frac{1}{i\hbar}[\lambda,\ h] = p_\lambda,$$

$$\dot{p}_\lambda = \frac{1}{i\hbar}[p_\lambda,\ h] = \frac{\gamma(\gamma+1)\hbar^2}{\lambda^3} - m^2\lambda\ -\ V_1'(\lambda). \tag{7.8}$$

We now introduce canonical coherent states for the ultralocal fields

$$|\,\psi\,\rangle = U[\psi]\,|\,0\,\rangle, \tag{7.9}$$

where, for each $\psi \in L^2$, the unitary transformation operator

$$U[\psi] = \exp\left\{\int d\lambda\, d^3x \left[\psi(x,\lambda)\, A^{\dagger}(x,\lambda) - \psi^{*}(x,\lambda) A(x,\lambda)\right]\right\}. \tag{7.10}$$

It is straightforward to show that

$$U^{\dagger}[\psi]\, A(x,\lambda)\, U[\psi] = A(x,\lambda) + \psi(x,\lambda),$$

$$U^{\dagger}[\psi]\, A^{\dagger}(x,\lambda)\, U[\psi] = A^{\dagger}(x,\lambda) + \psi^{*}(x,\lambda); \tag{7.11}$$

evidently the same relations hold for $B(x,\lambda)$ and $B^{\dagger}(x,\lambda)$ as well.

7.2. Selection of the Coherent States for Ultralocal Fields and the Classical Limit

Let us employ a Dirac-like (first quantized) formulation for functions of λ, and in particular let us set

$$\psi(x, \lambda) = (\lambda \mid \hbar^{-1/2}\, \alpha(x)), \tag{7.12}$$

where

$$(\lambda \mid \hbar^{-1/2}\, \alpha(x)) \equiv \left(\frac{1}{\pi\hbar}\right)^{1/4} \exp\left\{ -\frac{1}{2\hbar}\left[\lambda - \varphi(x)\right]^2 + \frac{i}{\hbar}\lambda\pi(x) + i\,\mu \right\} \tag{7.13}$$

$$\alpha(x) = \frac{1}{\sqrt{2}}\left[\varphi(x) + i\,\pi(x) \right]. \tag{7.14}$$

The expression (7.13) has the form of a conventional canonical coherent state, μ is an arbitrary phase, and $\mid \alpha(x)) = \exp[\alpha(x)\, a^{\dagger} - \alpha^{*}(x)\, a] \mid 0) = \widetilde{U}[\alpha(x)] \mid 0)$, is similar to the ordinary coherent state $\mid \alpha)$ [18,19] except that in the present case α is a function of x. a, $a^{\dagger}$ are the usual annihilation and creation operators, which are related to λ and p_{λ} by the relation

$$\begin{aligned} a &= \frac{1}{\sqrt{2\hbar}}(\lambda + i p_{\lambda}), & a^{\dagger} &= \frac{1}{\sqrt{2\hbar}}(\lambda - i p_{\lambda}), \\ \lambda &= \sqrt{\frac{\hbar}{2}}(a + a^{\dagger}), & p_{\lambda} &= i\sqrt{\frac{\hbar}{2}}(a^{\dagger} - a). \end{aligned} \tag{7.15}$$

Note well the appearance of $\hbar$ in these various expressions.

In terms of these expressions we have

$$\langle \psi \mid \Phi_r(x) \mid \psi \rangle = \langle 0 \mid U^{\dagger}[\psi]\, \Phi_r(x)\, U[\psi] \mid 0 \rangle$$

$$= \int d\lambda \, <0 \mid [B^{\dagger}(x,\lambda) + \psi^{*}(x,\lambda)] \, \lambda \, [B(x,\lambda) + \psi(x,\lambda)] \mid 0>$$

$$= \int d\lambda \, [c(\lambda) + \psi^{*}(x,\lambda)] \, \lambda \, [c(\lambda) + \psi(x,\lambda)]$$

Since $c(\lambda)$ always takes the form $c(\lambda) = \frac{1}{|\lambda|^{\gamma}} \exp\left\{ -\frac{1}{2\hbar} m\lambda^{2} - y_1(\lambda) \right\}$, it follows that $\lim_{\hbar \to 0} \int d\lambda \, c(\lambda) \, \lambda \, \psi(x,\lambda) = 0$. Consequently

$$\lim_{\hbar \to 0} <\psi \mid \Phi_r(x) \mid \psi> = \lim_{\hbar \to 0} \int d\lambda \; \psi^{*}(x,\lambda) \, \lambda \, \psi(x,\lambda)$$

$$= \lim_{\hbar \to 0} \int d\lambda (\hbar^{-1/2} \alpha(x) \mid \lambda) \, \lambda \, (\lambda \mid \hbar^{-1/2} \alpha(x))$$

$$= \lim_{\hbar \to 0} (\hbar^{-1/2} \alpha(x) \mid \lambda \mid \hbar^{-1/2} \alpha(x))$$

$$= \varphi(x). \qquad (7.16)$$

Before we calculate $<\psi \mid \Phi_r(x,t) \mid \psi>$, we need to prove several useful identities. By using

$$\tilde{U}^{\dagger}[\hbar^{-1/2} \alpha(x)] \, \lambda \, \tilde{U}[\hbar^{-1/2} \alpha(x)] = \lambda + \varphi(x) ,$$

$$\tilde{U}^{\dagger}[\hbar^{-1/2} \alpha(x)] \, p_{\lambda} \, \tilde{U}[\hbar^{-1/2} \alpha(x)] = p_{\lambda} + \pi(x),$$

we can calculate any arbitrary monomial in λ's and p_{λ} 's:

$$(\hbar^{-1/2}\,\alpha(x) \mid \lambda^{\theta_1} \cdots p_\lambda^{\theta_n} \mid \hbar^{-1/2}\,\alpha(x))$$

$$= (0 \mid (\lambda + \varphi(x))^{\theta_1} \cdots (p_\lambda + \pi(x))^{\theta_n} \mid 0)$$

Because of the prefactor $\sqrt{\hbar}$ in the definition of λ $[=\sqrt{\frac{\hbar}{2}}\,(a + a^\dagger)\,]$ and p_λ $[= i\sqrt{\frac{\hbar}{2}}(a^\dagger - a)]$, all the terms involving λ and p_λ will go to zero when we take the limit $\hbar \to 0$. Thus we have

$$\lim_{\hbar \to 0} (\hbar^{-1/2}\,\alpha(x) \mid \lambda^{\theta_1} \cdots p_\lambda^{\theta_n} \mid \hbar^{-1/2}\,\alpha(x)) = \varphi^{\theta_1}(x) \cdots \pi^{\theta_n}(x). \qquad (7.17)$$

Now with $h(\frac{\partial}{\partial\lambda}, \lambda) = h = -\frac{1}{2}\hbar^2\frac{\partial^2}{\partial\lambda^2} + V(\lambda)$, we calculate

$$\lambda(t) = \exp(\tfrac{i}{\hbar} t\, h)\, \lambda \exp(-\tfrac{i}{\hbar} t\, h)$$

$$= \lambda + \tfrac{i}{\hbar} t\, [h, \lambda] + \frac{1}{2!}\left(\tfrac{i}{\hbar} t\right)^2 [h, [h, \lambda]] + \frac{1}{3!}\left(\tfrac{i}{\hbar} t\right)^3 [h, [h, [h, \lambda]]]] + \cdots$$

$$= \lambda + t p_\lambda - \frac{1}{2!} t^2 V'(\lambda) - \frac{1}{3!}\frac{t^3}{2} [V''(\lambda)\, p_\lambda + p_\lambda V''(\lambda)] + \cdots. \qquad (7.18)$$

After an expectation in the coherent states $| \hbar^{-1/2}\,\alpha(x))$ and the limit $\hbar \to 0$, we obtain

$$\lim_{\hbar \to 0} (\hbar^{-1/2}\,\alpha(x) \mid \lambda(t) \mid \hbar^{-1/2}\,\alpha(x))$$

$$= \varphi(x) + t\,\pi(x) - \frac{1}{2!} t^2 V_0'[\varphi(x)] - \frac{1}{3!} t^3\, V_0''[\varphi(x)]\, \pi(x) + \cdots, \qquad (7.19)$$

where we have used (7.17), and introduced

$$V_0[\varphi(x, t)] = \lim_{\hbar \to 0} V[\varphi(x, t)]. \tag{7.20}$$

Notice that $V_0(\varphi)$ no longer depends on $\hbar$, but it does contain a vestige of $\hbar$ in the term $0 \cdot \varphi^{-2}(x,t)$.

We now prove that the result of (7.19) is just the solution $\varphi(x, t)$ of the classical canonical equations with the Hamiltonian

$$H_{cl} = \int d^3x \left[\frac{1}{2}\pi^2(x,t) + V_0[\varphi(x, t)]\right], \tag{7.21}$$

$$\dot{\varphi}(x, t) = \pi(x, t), \qquad \dot{\pi}(x, t) = - V_0'[\varphi(x, t)]. \tag{7.22}$$

With $\varphi(x, 0) = \varphi(x)$, $\pi(x, 0) = \pi(x)$, we can use a Taylor series expansion:

$$\varphi(x, t) = \varphi(x, 0) + t\,\dot{\varphi}(x, 0) + \frac{1}{2} t^2 \ddot{\varphi}(x, 0) + \frac{1}{3!} t^3 \frac{\partial^3 \varphi(x, 0)}{\partial t^3} + \cdots$$

$$= \varphi(x) + t\pi(x, 0) + \frac{t^2}{2!}\dot{\pi}(x, 0) + \frac{t^3}{3!}\ddot{\pi}(x, 0) + \cdots$$

$$= \varphi(x) + t\pi(x) - \frac{t^2}{2!}V_0'[\varphi(x)] - \frac{t^3}{3!}\frac{d}{dt}V_0'(\varphi(x, t))\,|_{t=0} + \cdots$$

$$= \varphi(x) + t\,\pi(x) - \frac{t^2}{2!}V_0'[\varphi(x)] - \frac{t^3}{3!}V_0''[\varphi(x)]\,\pi(x) + \cdots. \tag{7.23}$$

In the same way, we can prove that

$$\lim_{\hbar \to 0} (\hbar^{-1/2} \alpha(x) \mid p_\lambda(t) \mid \hbar^{-1/2} \alpha(x)) = \pi(x, t).$$

Thus

$$\lim_{\hbar \to 0} \langle \psi \mid \Phi_r(x, t) \mid \psi \rangle$$

$$= \lim_{\hbar \to 0} \langle \psi \mid \int d\lambda \, B^\dagger(x, \lambda) \, \lambda(t) \, B(x, \lambda) \mid \psi \rangle$$

$$= \lim_{\hbar \to 0} (\hbar^{-1/2} \alpha(x) \mid \lambda(t) \mid \hbar^{-1/2} \alpha(x)) = \varphi(x, t); \tag{7.24}$$

and evidently

$$\lim_{\hbar \to 0} \langle \psi \mid \dot{\Phi}_r(x, t) \mid \psi \rangle = \pi(x, t). \tag{7.25}$$

According to (7.22), $\varphi(x, t)$ satisfies the classical equation of motion, namely

$$\ddot{\varphi}(x, t) = - V_0'[\varphi(x, t)]. \tag{7.26}$$

Let us present two examples:

Example 1. Ultralocal Pseudo-Free Theory

The quantum Hamiltonian of the ultralocal pseudo-free field is given by

$$H = \int d^3x \int d\lambda \; B^\dagger(x, \lambda) \, h\left(\frac{\partial}{\partial \lambda}, \lambda\right) B(x, \lambda)$$

where

$$h(\frac{\partial}{\partial\lambda}, \lambda) = \frac{1}{2} p_\lambda^2 + \frac{\gamma(\gamma+1)\hbar^2}{2\lambda^2} + \frac{1}{2}m^2\lambda^2 + m(\gamma - 1/2)\hbar .$$

The model function $c(\lambda)$ for the system is determined through (6.7b), and the result is

$$c(\lambda) = \frac{1}{|\lambda|^\gamma} \exp\left\{ - \frac{1}{2\hbar} m\lambda^2 \right\}. \qquad (7.27)$$

Expressing the quantum Hamiltonian in terms of the renormalized field operators, we have

$$H = \int \left\{ \frac{1}{2}\Pi_r^2(x) + \frac{1}{2}m^2\Phi_r^2(x) \right\} d^3x$$

where we have defined the local operator

$$\Pi_r^2 = \dot{\Phi}_r^2 + \gamma(\gamma+1)\hbar^2 \Phi_r^{-2} + m(\gamma - \frac{1}{2})\hbar \Phi_r^0 .$$

Obviously this definition means that Π_r neither fits into the canonical framework nor fulfills the commutation relation $[\Phi_r(x) , \Pi_r(y)] = i\hbar\delta(x - y)$.

From the above analysis, the classical limit of the field operators in the coherent states denoted by $\varphi(x, t)$ and $\pi(x,t)$ fits into the classical canonical equation (7.22), with $V_I(\varphi) = 0$, and $\varphi(x, t)$ is the solution of the classical pseudo-free field equation of motion (7.26)[20] :

$$\ddot{\varphi}(x, t) = 0\cdot\varphi^{-3}(x,t) - m^2\varphi(x,t) ,$$

where the role of $0\cdot\varphi^{-3}(x,t)$ is to assure that $\varphi(x, t) > 0$ or $\varphi(x, t) < 0$, namely, that $\varphi(x, t)$ is not allowed to cross through $\varphi(x, t) = 0$, This kind of solution can be secured if we replace the above equation by the scale-covariant equation of motion[3]

$$\varphi(x,t)\left[\ddot{\varphi}(x, t) + m^2\varphi(x, t)\right] = 0 .$$

For readers interested in the equation of motion of the pseudo-free field operators, Ref. 21 is recommended.

Example 2. Ultralocal Field with Interaction Potential $g\varphi^4 + \frac{g^2}{2m^2}\varphi^6$

The quantum Hamiltonian is $H = \int d^3x \int d\lambda\ B^{\dagger}(x, \lambda)\, h(\frac{\partial}{\partial\lambda}, \lambda)\, B(x, \lambda)$,

and the expressions of $h(\frac{\partial}{\partial\lambda}, \lambda)$ and $c(\lambda)$ for such a model are

$$h(\frac{\partial}{\partial\lambda}, \lambda) = \frac{1}{2} p_\lambda^2 + \frac{\gamma(\gamma+1)\hbar^2}{2\lambda^2} + \frac{1}{2}m^2\lambda^2 + g\lambda^4 + \frac{g^2}{2m^2}\lambda^6 + m\,(\gamma - 1/2)\hbar + \frac{g\,(\gamma - 3/2)}{m}\lambda^2\hbar$$

,

$$c(\lambda) = \frac{1}{|\lambda|^{\gamma}} \exp\left\{ -\frac{1}{2\hbar}m\lambda^2 - \frac{g}{4m\hbar}\lambda^4 \right\}. \qquad (7.28)$$

The Hamiltonian expressed in terms of the renormalized field operators is

$$H = \int \left\{ \frac{1}{2}\Pi_r^2(x) + \frac{1}{2}\left[m^2 + \frac{2g\,(\gamma - 3/2)\hbar}{m}\right]\Phi_r^2(x) + g\Phi_r^4(x) + \frac{g^2}{2\,m^2}\Phi_r^6(x) \right\} d^3x ,$$

with $\Pi_r^2 = \dot{\Phi}_r^2 + \gamma(\gamma+1)\hbar^2\,\Phi_r^{-2} + m(\gamma - \frac{1}{2})\hbar\,\Phi_r^0$. Since the last two terms in $h(\frac{\partial}{\partial\lambda}, \lambda)$ vanish when we take the classical limit, according to (7.20), the corresponding classical Hamiltonian when $\hbar \to 0$ is

$$H_{cl} = \int \left\{ \frac{1}{2}\pi^2(x) + 0\cdot\varphi^{-2}(x) + \frac{1}{2}m^2\varphi^2(x) + g\varphi^4(x) + \frac{g^2}{2\,m^2}\varphi^6(x) \right\} d^3x ,$$

The canonical equations are

$$\dot{\varphi}(x,t) = \pi(x, t)$$

$$\dot{\pi}(x, t) = 0\cdot\varphi^{-3}(x,t) - m^2\varphi(x,t) - 4g\varphi^3(x,t) - \frac{3g^2}{m^2}\varphi^5(x,t) \qquad (7.29)$$

and the equation of motion is

$$\ddot{\varphi}(x, t) = 0\cdot\varphi^{-3}(x,t) - m^2\varphi(x,t) - 4g\varphi^3(x,t) - \frac{3g^2}{m^2}\varphi^5(x,t) ,$$

according to (7.26).

Again the role of $0\cdot\varphi^{-3}(x,t)$ is to assure that $\varphi(x, t) > 0$ or $\varphi(x, t) < 0$, namely, that $\varphi(x, t)$ is not allowed to cross through $\varphi(x, t) = 0$, This kind of solution can be secured if we replace (7.29) by the scale-covariant equation of motion[3]

$$\varphi(\mathbf{x},t)\ddot{\varphi}(\mathbf{x},t) = -m^2\varphi^2(\mathbf{x},t) - 4g\varphi^4(\mathbf{x},t) - \frac{3g^2}{m^2}\varphi^6(\mathbf{x},t)$$

which is satisfied by the classical limit of the ultralocal field $\lim_{\hbar \to 0} \langle \psi | \Phi_r(\mathbf{x}, t) | \psi \rangle$, according to Eq. (7.24).

CHAPTER 8
PATH-INTEGRAL FORMULATION OF ULTRALOCAL MODELS

In an earlier chapter, we have pointed out that the path-integral of the quartic self-coupled ultralocal scalar field leads to an trivial (Gaussian) result when approached by the conventional lattice limit (Chap. 5). However, the characterization of infinitely divisible distributions allows ultralocal models alternative operator solutions (Chap. 6) which suggest a new expression of path-integral with a nonvanishing, nonclassical potential. In this chapter, we seek to replace the standard lattice formulation by a nonstandard one which admits nontrivial results. Specifically, we will use these alternative, nonperturbative operator solutions to construct nontrivial lattice-space path integrals for nonrenormalizable ultralocal models. As we will see, the indefinite, nonclassical, singular potential required for the nontriviality in multi-component case has different effects on distributions compared to the single-component case, however the essential property of reweighting the distribution at the origin is similar. The appearance of additional nonclasssical, singular potentials suggests that we can not always place the classical Lagrangian or classical Hamiltonian directly into the path-integral formulation, or in other words, a straightforward canonical quantization of fields with infinite degrees of freedom does not always apply.

8.1. Operator Solutions of Ultralocal Models

In this section, we will select certain results, which will be used here, from Chapter 6 on the operator analysis of ultralocal models[15], and put them into completely proper dimensional forms. Following Chapter 6, the N -component ultralocal field operator can be expressed as

$$\vec{\Phi}(x) = \int d\vec{\lambda}\, B^{\dagger}(x, \vec{\lambda})\, \vec{\lambda}\, B(x, \vec{\lambda}) \tag{8.1}$$

Where the operators $B^{\dagger}(x, \vec{\lambda})$ and $B(x, \vec{\lambda})$ are translated Fock operators, defined through $B(x, \vec{\lambda}) = A(x, \vec{\lambda}) + c(\lambda)$, $\lambda \equiv \sqrt{\vec{\lambda}^2}$, while $A^{\dagger}(x, \vec{\lambda})$ and $A(x, \vec{\lambda})$ denote conventional irreducible Fock representation operators for which $|0>$ is the unique vacuum, $A(x, \vec{\lambda})|0> = 0$. for all $x \in \Re^{n-1}$, $\vec{\lambda} \in \Re^{N}$. The only nonvanishing commutator is given by

$$\left[A(x, \vec{\lambda}) , A^{\dagger}(x', \vec{\lambda}')\right] = \delta(x-x')\, \delta(\vec{\lambda}-\vec{\lambda}'). \tag{8.2}$$

Obviously, the operators $B^{\dagger}(x, \vec{\lambda})$ and $B(x, \vec{\lambda})$ follow the same commutation relation (8.2). For the single-component case, (8.1) becomes

$$\Phi(x) = \int d\lambda\, B^{\dagger}(x, \lambda)\, \lambda\, B(x, \lambda). \tag{8.3}$$

Renormalized products follow from the operator product expansion:

$$\Phi_i(x)\, \Phi_j(y) = \delta(x-y) \int d\vec{\lambda}\, B^{\dagger}(x, \vec{\lambda})\, \lambda_i \lambda_j\, B(x, \vec{\lambda}) + : \Phi_i(x)\, \Phi_j(y) : , \quad i, j = 1, 2, \cdots N$$

which suggests the definition of the renormalized square as

$$\Phi^2_{rij}(x) \equiv b\int d\vec{\lambda}\, B^\dagger(x,\vec{\lambda})\,\lambda_i\lambda_j\, B(x,\vec{\lambda}) \equiv Z\,\Phi_i(x)\,\Phi_j(y)\,, \quad Z = b/\delta\,(0). \tag{8.4}$$

Here we have introduced an auxiliary constant b designed to keep Z dimensionless. (In the previous chapters, we have generally chosen b=1). With $[\cdot]$ ≡ dimension $(\cdot)$, we have $[b]=[x]^{1-n}$. More generally, the renormalized local powers of the field are given by

$$\Phi^k_{r\,i_1\cdots i_k}(x) \equiv b^{k-1}\int d\vec{\lambda}\, B^\dagger(x,\vec{\lambda})\,\lambda_{i_1}\cdots\lambda_{i_k}\, B(x,\vec{\lambda}) \equiv Z^{k-1}\Phi_{i_1}\cdots\Phi_{i_k} \tag{8.5}$$

$$i_1,\cdots,i_k = 1,2,3,\cdots N \text{ and } k = 1,2,3,\cdots.$$

Occasionally, we need to use the renormalized negative powers of the field even though they are not local operators. Take the one-component case for example:

$$\Phi^k_{ri}(x) \equiv \frac{1}{b}\int d\vec{\lambda}B^\dagger(x,\vec{\lambda})\,(b\lambda_i)^k B(x,\vec{\lambda}) \equiv \frac{1}{Z}[Z\,\Phi_i(x)]^k, \quad i = 1,2\cdots N; \tag{8.6}$$

it can formally be extended to $k \le 0$, as long as we define

$$\Phi^{-1}_i(x) \equiv Z^2\int d\vec{\lambda}B^\dagger(x,\vec{\lambda})\frac{1}{b^2\lambda_i}B(x,\vec{\lambda}) \text{ and } \int d\vec{\lambda}B^\dagger(x,\vec{\lambda})\frac{1}{b^2\lambda_i}B(x,\vec{\lambda}) \equiv \Phi^{-1}_{ri}(x).$$

The Hamiltonian of $O(N)$ invariant, N-component ultralocal fields can be expressed as

$$H = \int dx \int d\vec{\lambda}\ B^{\dagger}(x, \vec{\lambda})\, h(\overrightarrow{\nabla_{\lambda}}, \vec{\lambda})\, B(x, \vec{\lambda})$$

$$= \int dx \int d\vec{\lambda}\, A^{\dagger}(x, \vec{\lambda})\, h(\overrightarrow{\nabla_{\lambda}}, \vec{\lambda})\, A(x, \vec{\lambda}). \qquad (8.7)$$

Where $h(\overrightarrow{\nabla_{\lambda}}, \vec{\lambda}) = -\frac{1}{2b}\hbar^2\nabla_{\lambda}^2 + V(\lambda)$ is a self-adjoint operator in the $\vec{\lambda}$ variable alone. The connection between these two relations holds when we insist that $h(\overrightarrow{\nabla_{\lambda}}, \vec{\lambda})\, c(\lambda) = 0$, and in order for $|0>$ to be the unique ground state of H, it follows that $c(\lambda) \notin \mathcal{L}_2(\Re^N)$. Since $[c]^2 = [x]^{1-n}[\lambda]^{-N}$ and $[\lambda]^2 = [x]^n[\hbar]$, we have $[c]^2 = [x]^{1-n(1+N/2)}[\hbar]^{-N/2}$. Therefore, by taking into account of the dimensions,, the model function may be expressed as

$$c(\lambda) = \hbar^{(2\Gamma-N)/4} b^{\alpha} \frac{1}{\lambda^{\Gamma}} \exp\left\{ -\frac{(mb)}{2\hbar}\lambda^2 - y(\lambda, b, \hbar)\right\}, \qquad (8.8)$$

where $\alpha = [\, n(1-\Gamma+N/2)-1\,]\,/2(n-1)$. With a proper account of dimensions, the self-adjoint base Hamiltonian obeys

$$h(\overrightarrow{\nabla_{\lambda}}, \vec{\lambda}) = -\frac{1}{2b}\hbar^2\nabla_{\lambda}^2 + \frac{\Gamma(\Gamma+2-N)\hbar^2}{2b\lambda^2} + m\hbar(\Gamma-N/2) + \frac{1}{2}m^2 b\lambda^2 + \frac{1}{b}V_I(b\lambda)\,. \qquad (8.9)$$

Obviously, For the single-component case, (8.8) and (8.9) become

$$c(\lambda) = \hbar^{(2\gamma-1)/4} b^{[n(3/2-\gamma)-1]/2(n-1)} \frac{1}{|\lambda|^{\gamma}} \exp\left\{ -\frac{b}{2\hbar}m\lambda^2 - y(\lambda, b, \hbar)\right\}. \qquad (8.10)$$

$$h(\partial/\partial\lambda, \lambda) = -\frac{1}{2b}\hbar^2\frac{\partial^2}{\partial\lambda^2} + \frac{\gamma(\gamma+1)\hbar^2}{2b\lambda^2} + m\hbar(\gamma - 1/2) + \frac{1}{2}m^2b\lambda^2 + \frac{1}{b}V_I(b\lambda) \ . \qquad (8.11)$$

The Hamiltonian may also be expressed in terms of renormalized fields as

$$H = \int \left\{\frac{1}{2}\vec{\Pi}_r^2(x) + \frac{1}{2}m^2\vec{\Phi}_r^2(x) + V_I[\vec{\Phi}_r(x)]\right\}dx, \qquad (8.12)$$

where $\vec{\Pi}_r^2(x) = \dot{\vec{\Phi}}_r^2(x) + \Gamma(\Gamma+2 - N)\hbar^2b^2\,|\,\vec{\Phi}_r(x)\,|^{-2} + m\hbar\,(2\Gamma - N)b\,|\,\vec{\Phi}_r(x)\,|^{0}$, and in terms of bare fields as

$$H = \int \left\{\frac{1}{2}Z\vec{\Pi}^2(x) + \frac{1}{2}m^2Z\vec{\Phi}^2(x) + \frac{1}{Z}V_I[Z\vec{\Phi}(x)]\right\}dx \qquad (8.13)$$

where $Z\vec{\Pi}^2(x) \equiv Z\dot{\vec{\Phi}}^2(x) + \Gamma(\Gamma+2-N)\hbar^2b^2Z^{-3}\,|\,\vec{\Phi}(x)\,|^{-2} + m\hbar\,(2\Gamma-N)bZ^{-1}\,|\,\vec{\Phi}(x)\,|^{0} = \vec{\Pi}_r^2(x)$.

For the single-component case, the Hamiltonian becomes

$$H = \int \left\{\frac{1}{2}\Pi_r^2(x) + \frac{1}{2}m^2\Phi_r^2(x) + V_I[\Phi_r(x)]\right\}dx, \qquad (8.14)$$

$$= \int \left\{\frac{1}{2}Z\Pi^2(x) + \frac{1}{2}m^2Z\Phi^2(x) + \frac{1}{Z}V_I[Z\Phi_r(x)]\right\}dx \qquad (8.15)$$

where $\Pi_r^2 = \dot{\Phi}_r^2 + \gamma(\gamma+1)\hbar^2b^2\,\Phi_r^{-2} + m\hbar(2\gamma - 1)b\,\Phi_r^0$ and $Z\Pi^2 \equiv Z\dot{\Phi}^2 + \gamma(\gamma+1)\hbar^2b^2\Phi^{-2}Z^{-3}$ $+ m\hbar(2\gamma - 1)b\,\Phi^0Z^{-1} = \Pi_r^2$.

The Hamiltonian operator (8.7) readily leads to the Heisenberg field operator which is given by

$$\vec{\Phi}(\mathbf{x},t) = e^{iHt/\hbar}\,\vec{\Phi}(\mathbf{x})\,e^{-iHt/\hbar} = \int d\vec{\lambda}\, B^{\dagger}(\mathbf{x},\vec{\lambda})\, e^{iht/\hbar\vec{\lambda}} e^{-iht/\hbar}\, B(\mathbf{x},\vec{\lambda})$$

$$= \int d\vec{\lambda}\, B^{\dagger}(\mathbf{x},\vec{\lambda})\,\vec{\lambda}(t)\, B(\mathbf{x},\vec{\lambda}) \qquad (8.16)$$

After this partial review, we turn our attention to the path-integral formulation of ultralocal models.

8.2 Euclidean-Space Path-Integral Formulation of Single-Component Ultralocal Fields

In this section, we will start from the operator solution presented in §8.1 to derive the Euclidean time lattice-space formulation. Attention is focused on the time-ordered vacuum to vacuum expectation functional

$$C[f] \equiv < 0 \mid \mathrm{Texp}\left[\frac{i}{\hbar}\int dx\, dt\, \Phi(x,t)\, f(x,t)\right] \mid 0 >. \tag{8.17}$$

Insert (8.3), then we have

$$C[f] = < 0 \mid \mathrm{Texp}\left[\frac{i}{\hbar}\int dx\, dt \int d\lambda\, B^{\dagger}(x,\lambda)\, f(x,t)\lambda(t)\, B(x,\lambda)\right] \mid 0 >,$$

By using the normal ordering technique, C[f] becomes

$$< 0 \mid T{:}\, \exp\left[\int dx \int d\lambda\, B^{\dagger}(x,\lambda)\, \{ \exp[\frac{i}{\hbar}\int dt\, f(x,t)\lambda(t)] - 1 \}\, B(x,\lambda)\right] {:} \mid 0 >.$$

Notice that only the part inside { } depends on time t. Therefore we can move the time-ordering sign T into the bracket, so we find that

$$C[f] = < 0 \mid {:}\, \exp\left[\int dx \int d\lambda\, B^{\dagger}(x,\lambda)\, \{ \mathrm{Texp}[\frac{i}{\hbar}\int dt\, f(x,t)\lambda(t)] - 1 \}\, B(x,\lambda)\right] {:} \mid 0 >$$

$$= \exp\left[\int dx \int d\lambda\, c(\lambda)\, \{ \mathrm{Texp}[\frac{i}{\hbar}\int dt\, f(x,t)\lambda(t)] - 1\}\, c(\lambda)\right]$$

$$= \exp\left[\int dx \int d\lambda\, c(\lambda)\, \{ \mathrm{Texp}[\int dt\, (\tfrac{i}{\hbar} f(x,t)\lambda - h(\partial/\partial\lambda, \lambda))\,] - 1\}\, c(\lambda)\right]$$

$$= \lim_{a\to 0} \exp\left[\sum_k a \int d\lambda\, c_\delta(\lambda)\, \{ \mathrm{Texp}[\int dt\, (\tfrac{i}{\hbar} f_k(x,t)\lambda - h_\delta(\partial/\partial\lambda, \lambda))\,] - 1\}\, c_\delta(\lambda)\right].$$

Here in the last step we have employed a lattice-space regularization in which a is the space volume. Let

$$c_\delta(\lambda) = \hbar^{(2\gamma-1)/4}\, b^\alpha\, (\lambda^2 + \delta^2)^{-\gamma/2} \exp\left\{ -\frac{b}{2\hbar} m(\lambda^2+\delta^2) - y(\lambda, \delta)\right\},$$

where δ is the function of a, $\alpha = [n(3/2-\gamma)-1]/2(n-1)$, and choose δ so that $a\int d\lambda\, c_\delta^2(\lambda) = 1$. Then we find out that $\delta = \sqrt{\hbar}\, (a\, b^{2\alpha} F)^{1/(2\gamma-1)}$, $F = \int_{-\infty}^{\infty} dx\, (x^2+1)^{-\gamma}$ for $\gamma > 1/2$; and $\delta \propto \sqrt{\hbar}\, b^{n/[2(1-n)]} \exp[-1/(2ab)]$ for $\gamma = 1/2$. Therefore $\delta \to 0$, as $a\to 0$, $\lim_{a\to 0} c_\delta(\lambda) = c(\lambda)$, and

$$C[f] = \lim_{a\to 0} \prod_k \exp\left[a \int d\lambda\, c_\delta(\lambda)\, \{ \mathrm{Texp}[\int dt\, (\tfrac{i}{\hbar} f_k(x,t)\lambda - h_\delta(\partial/\partial\lambda, \lambda))\,] - 1\}\, c_\delta(\lambda)\right]$$

$$= \lim_{a\to 0} \prod_k a \int d\lambda\, c_\delta(\lambda)\, \{ \mathrm{Texp}[\int dt\, \tfrac{i}{\hbar} f_k(t)\lambda - \int dt\, h_\delta(\partial/\partial\lambda, \lambda)\,]\}\, c_\delta(\lambda)$$

where $a^{1/2} c_\delta(\lambda)$ represents the normalized ground state of $h_\delta(\partial/\partial\lambda, \lambda)$, i.e., $h_\delta(\partial/\partial\lambda, \lambda)\, c_\delta(\lambda) = 0$. To the required accuracy in δ, we have

$$h_\delta(\partial/\partial\lambda, \lambda) = -\frac{1}{2b}\hbar^2 \frac{\partial^2}{\partial\lambda^2} + \frac{\gamma(\gamma+1)\lambda^2 - \gamma\delta^2}{2b(\lambda^2+\delta^2)^2}\hbar^2 + m(\gamma-1/2)\hbar + \frac{1}{2}m^2 b\lambda^2 + \frac{1}{b}V_I(b\lambda)$$

and C[f] is reduced to a path-integral form of single degree quantum mechanics as

$$\lim_{a\to 0}\prod_k \mathcal{N}\int D\lambda \exp\int dt \left\{\frac{i}{\hbar}f_k\lambda - \left[\frac{1}{2}b\dot{\lambda}^2 + \frac{\gamma(\gamma+1)\lambda^2 - \gamma\delta^2}{2b(\lambda^2+\delta^2)^2}\hbar^2 + m(\gamma-1/2)\hbar\right.\right.$$

$$\left.\left. + \frac{1}{2}m^2b\lambda^2 + \frac{1}{b}V_I(b\lambda)\right]\right\} \qquad (8.18)$$

$$= \lim_{a\to 0}\mathcal{N}\int (\prod D\lambda_k)\exp\sum_k \int dt\left\{\frac{i}{\hbar}f_k\lambda_k - \left[\frac{1}{2}b\dot{\lambda}_k^2 + \frac{\gamma(\gamma+1)\lambda_k^2 - \gamma\delta^2}{2b(\lambda_k^2+\delta^2)^2}\hbar^2 + m(\gamma-1/2)\hbar\right.\right.$$

$$\left.\left. + \frac{1}{2}m^2b\lambda_k^2 + \frac{1}{b}V_I(b\lambda_k)\right]\right\}$$

Set $\lambda_k = a\Phi_k$, then this expression becomes

$$C[f] = \lim_{a\to 0}\mathcal{N}\int (\prod D\Phi_k)\exp\sum_k a\int dt\left\{\frac{i}{\hbar}f_k\Phi_k - \left[\frac{1}{2}ab\dot{\Phi}_k^2 + \frac{\gamma(\gamma+1)\Phi_k^2 - a^{-2}\gamma\delta^2}{2b(\Phi_k^2+a^{-2}\delta^2)^2}\frac{\hbar^2}{a^3}\right.\right.$$

$$\left.\left. + m\hbar(\gamma-1/2)\frac{1}{a} + \frac{1}{2}m^2ab\Phi_k^2 + \frac{1}{ab}V_I(ab\Phi_k)\right]\right\} \qquad (8.19)$$

In the formal limit $a\to 0$, we have

$$C[f] = \mathcal{N}\int D\Phi \exp\int dx\, dt\left\{\frac{i}{\hbar}f(x,t)\Phi(x,t) - \left[\frac{1}{2}Z\dot{\Phi}^2 + \frac{\gamma(\gamma+1)\hbar^2b^2}{2}Z^{-3}\Phi^{-2}\right.\right.$$

$$\left.\left. + m\hbar(\gamma-1/2)b\,\Phi^0Z^{-1} + \frac{1}{2}m^2Z\Phi^2 + \frac{1}{Z}V_I(Z\Phi)\right]\right\}$$

$$= \mathcal{N}\int D\Phi \exp\left\{\frac{i}{\hbar}\int dx\, dt\, f(x,t)\Phi(x,t) - \int dt\, L\right\} \tag{8.20}$$

Where

$$L = \int dx \left\{\frac{1}{2}Z\dot{\Phi}^2 + \frac{\gamma(\gamma+1)\hbar^2 b^2}{2}Z^{-3}\Phi^{-2} + m\hbar(\gamma - 1/2)b\,\Phi^0 Z^{-1} + \frac{1}{2}m^2 Z\Phi^2 + \frac{1}{Z}V_I(Z\Phi)\right\},$$

and notice that we have used the formal factor $Z = b/\delta(0) = \lim_{a\to 0} ba$.

Obviously, because of the appearance of the additional nonclasssical, singular potential, the form of L is no longer the classical Lagrangian of the ultralocal model which reads $L_{cl} = \int dx \left\{\frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}m^2\Phi^2 + V_I(\Phi)\right\}$. Such a difference suggests that we can not always place the classical Lagrangian or classical Hamiltonian directly into the path-integral formulation, or in other words, a straightforward canonical quantization of fields with infinitely many degrees of freedom does not always hold.

8.3 Euclidean Space Path-Integral Formulation of Multi-Component Ultralocal Fields

Following what we did in §8.2 and using the results presented in §8.1, we have the time-ordered vacuum-to-vacuum expectation functional

$$C[f] \equiv < 0 \mid \mathrm{Texp}\left[\frac{i}{\hbar}\int dx\, dt\, \vec{\Phi}(x,t)\cdot\vec{f}(x,t)\right] \mid 0 >. \tag{8.21}$$

Substitute (8.1) into the above equation, and it becomes

$$= < 0 \mid \mathrm{Texp}\left[\frac{i}{\hbar}\int dx\, dt \int d\vec{\lambda}\, B^{\dagger}(x,\vec{\lambda})\, \vec{f}(x,t)\cdot\vec{\lambda}(t)\; B(x,\vec{\lambda})\right] \mid 0 >$$

As we did before, by using the normal ordering technique, we obtain

$$C[f] = < 0 \mid T{:}\exp\left[\int dx \int d\vec{\lambda}\, B^{\dagger}(x,\vec{\lambda})\, \{\exp[\int dt\, \frac{i}{\hbar}\vec{f}(x,t)\cdot\vec{\lambda}(t)] - 1\}\, B(x,\vec{\lambda})\right] {:} \mid 0 >$$

$$= < 0 \mid {:}\exp\left[\int dx \int d\vec{\lambda}\, B^{\dagger}(x,\vec{\lambda})\, \{\mathrm{Texp}[\int dt\, \frac{i}{\hbar}\vec{f}(x,t)\cdot\vec{\lambda}(t)] - 1\}\, B(x,\vec{\lambda})\right] {:} \mid 0 >$$

$$= \exp\left[\int dx \int d\vec{\lambda}\, c(\lambda)\, \{\mathrm{Texp}[\int dt\, \frac{i}{\hbar}\vec{f}(x,t)\cdot\vec{\lambda}(t)] - 1\}\, c(\lambda)\right]$$

$$= \exp\left[\int dx \int d\vec{\lambda}\, c(\lambda)\, \{\mathrm{Texp}[\int dt\, (\frac{i}{\hbar}\vec{f}(x,t)\cdot\vec{\lambda} - h(\overrightarrow{\nabla_{\lambda}},\vec{\lambda}))] - 1\}\, c(\lambda)\right]$$

$$= \lim_{a\to 0} \exp\left[\sum_{k} a \int d\vec{\lambda}\, c_{\delta}(\lambda)\, \{\mathrm{Texp}[\int dt\, (\frac{i}{\hbar}\vec{f}_{k}(x,t)\cdot\vec{\lambda} - h_{\delta}(\overrightarrow{\nabla_{\lambda}},\vec{\lambda}))] - 1\}\, c_{\delta}(\lambda)\right]$$

Where in the last step we have employed a lattice-space regularization in which a is the lattice-cell volume. Let

$$c_\delta(\lambda) = \hbar^{(2\Gamma-N)/4} b^{\alpha} \frac{1}{(\lambda^2+\delta^2)^{\Gamma/2}} \exp\left\{ -\frac{(mb)}{2\hbar}(\lambda^2+\delta^2) - y(\lambda,\delta)\right\},$$

where $\alpha = [\, n(1-\Gamma+N/2)-1\,]\,/2(n-1)$, and choose $a\int d\vec{\lambda}\, c_\delta^2(\lambda) = 1$, then we have

$$C[f] = \lim_{a\to 0}\prod_k \exp\left[a\int d\vec{\lambda}\; c_\delta(\lambda)\,\{\,\mathrm{Texp}[\int dt\,(\tfrac{i}{\hbar}\vec{f}_k(x,t)\cdot\vec{\lambda} - h_\delta(\vec{\nabla}_\lambda,\vec{\lambda}))\,] - 1\}\, c_\delta(\lambda)\right]$$

$$= \lim_{a\to 0}\prod_k a\int d\vec{\lambda}\, c_\delta(\lambda)\,\{\,\mathrm{Texp}[\int dt\,(\tfrac{i}{\hbar}\vec{f}_k(x,t)\cdot\vec{\lambda} - h_\delta(\vec{\nabla}_\lambda,\vec{\lambda}))\,]\,\}\, c_\delta(\lambda),$$

where $a^{1/2}c_\delta(\lambda)$ denotes the normalized ground state of $h_\delta(\vec{\nabla}_\lambda,\vec{\lambda})$, i.e., $h_\delta(\vec{\nabla}_\lambda,\vec{\lambda})\, c_\delta(\lambda) = 0$. Thus, to the required accuracy in δ, we have

$$h_\delta(\vec{\nabla}_\lambda,\vec{\lambda}) = -\frac{1}{2b}\hbar^2\nabla_\lambda^2 + \frac{\Gamma(\Gamma+2-N)\hbar^2}{2b(\lambda^2+\delta^2)} - \frac{\Gamma(\Gamma+2)\hbar^2\delta^2}{2b(\lambda^2+\delta^2)^2} + m\hbar(\Gamma-N/2) + \frac{1}{2}m^2b\lambda^2 + \frac{1}{b}V_I(b\lambda).$$

Expressing C[f] in a path-integral form, we get

$$C[f] = \lim_{a\to 0}\prod_k \mathcal{N}\int D\vec{\lambda}\exp\left[\frac{i}{\hbar}\int dt\,\vec{f}_k(t)\cdot\vec{\lambda} - \int dt\, L_\lambda(\vec{\lambda},\dot{\vec{\lambda}})\right], \qquad (8.22)$$

where

$$L_\lambda(\vec{\lambda}, \dot{\vec{\lambda}}) = \frac{1}{2} b\dot{\vec{\lambda}}^2 + \frac{\Gamma(\Gamma+2-N)\hbar^2}{2b(\lambda^2+\delta^2)} - \frac{\Gamma(\Gamma+2)\hbar^2\delta^2}{2b(\lambda^2+\delta^2)^2} + m\hbar(\Gamma - \frac{N}{2}) + \frac{1}{2}m^2 b\lambda^2 + \frac{1}{b}V_I(b\lambda) \ .$$

It is worth notice that in the cases of vanishing nonclassical, singular potential $\frac{\Gamma(\Gamma+2-N)\hbar^2}{2b(\lambda^2+\delta^2)}$, another regularized singular potential, $-\frac{\Gamma(\Gamma+2)\hbar^2\delta^2}{2b(\lambda^2+\delta^2)^2}$, will take over the reweighting role of the distribution, but only at the origin $\lambda = 0$. Unlike the regularized term $\frac{\gamma(\gamma+1)\lambda^2 - \gamma\delta^2}{2b(\lambda^2+\delta^2)^2}\hbar^2$ in (8.18), which is reduced as $a \to 0$ to the positive nonclassical potential $\frac{\gamma(\gamma+1)\hbar^2}{2b\lambda^2}$ when $\lambda \neq 0$, the second term in the expression above may be reduced to either a positive, zero or negative nonclassical potential $\frac{\Gamma(\Gamma+2-N)\hbar^2}{2b\lambda^2}$ when $\lambda \neq 0$ (the third term reduces to zero), so we can expect that different effects of reweighting the distribution may happen. But at the point $\lambda = 0$ the combination of above two terms gives a regularized negative singular potential $-\frac{\Gamma(\Gamma+2)\hbar^2}{2b\,\delta^2}$, which is similar to $\frac{-\gamma\hbar^2}{2b\delta^2}$ of $N = 1$ case. Therefore the reweighting behavior near $\lambda = 0$ is essentially the same.

Since the subscript k denotes independent spatial points, we therefore may write (8.22) as

$$C[f] = \lim_{a\to 0} \mathcal{N} \int (\prod D\vec{\lambda}_k) \exp\left[\sum_k \int dt \left\{\frac{i}{\hbar}\vec{f}_k \cdot \vec{\lambda}_k - L_\lambda(\vec{\lambda}_k, \dot{\vec{\lambda}}_k)\right\}\right] .$$

Set $\lambda_k = a\Phi_k$ and take the limit, we then obtain

$$C[f] = \lim_{a\to 0} \mathcal{N} \int (\prod D\vec{\Phi}_k) \exp\left[\sum_k a \int dt \left\{\frac{i}{\hbar} \vec{f}_k \cdot \vec{\Phi}_k - \frac{1}{a} L_\lambda(a\vec{\Phi}_k, a\dot{\vec{\Phi}}_k)\right\}\right]$$

$$= \mathcal{N} \int D\vec{\Phi} \exp\left[\int dx \int dt \frac{i}{\hbar} \vec{f}(x, t)\cdot\vec{\Phi}(x, t) - \int dt\, L_\Phi(\vec{\Phi}, \dot{\vec{\Phi}})\right], \qquad (4.3)$$

where we have used the formal factor $Z = b/\delta(0) = \lim_{a\to 0} ba$, and

$$L(\vec{\Phi}, \dot{\vec{\Phi}}) \equiv \lim_{a\to 0} \frac{1}{a} L_\lambda(a\vec{\Phi}_k, a\dot{\vec{\Phi}}_k)$$

$$= \int dx \left\{\frac{1}{2} Z\dot{\vec{\Phi}}^2 + \frac{1}{2}\Gamma(\Gamma+2-N)\hbar^2 b^2 Z^{-3} |\vec{\Phi}|^{-2} + m\hbar(\Gamma - \frac{N}{2}) bZ^{-1} |\vec{\Phi}|^0 + \frac{1}{2} m^2 Z\vec{\Phi}^2 + \frac{1}{Z} V_I(Z\vec{\Phi})\right\}$$

Once again, the appearance of the additional nonclasssical, singular potential, has made the form of L no longer the classical Lagrangian of ultralocal model which reads $L_{cl} = \int \left\{\frac{1}{2}\dot{\vec{\Phi}}^2(x) + \frac{1}{2} m^2 \vec{\Phi}^2(x) + V_I[\vec{\Phi}(x)]\right\} dx$. Such a difference suggests again that we can not always set the classical Lagrangian or classical Hamiltonian directly in a path-integral formulation, or in other words, the canonical quantization for multi-component fields with infinitely many degrees of freedom does not always hold.

We are unable to illustrate a path integral construction for $N = \infty$ directly, although there is always the option to introduce $N \to \infty$ outside of (4.3). On the other hand, the desired generating function for $N = \infty$ can be formulated in the operator language. In particular, by using the results of the

operator solution in Ref. 7 for the infinite-component case (i.e., (3.2) through (3.7) in Ref. 7), we have

$$C[f] \equiv <0\ |\mathrm{T}\exp\left[\frac{i}{\hbar}\int dx\,dt\,\vec{\Phi}(x,t)\cdot\vec{f}(x,t)\right]|\ 0>$$

$$= <0\ |\mathrm{T}\exp\left[\frac{i}{\hbar}\int dx\,dt\int d\rho(\vec{\lambda})\,B^{\dagger}(x,\vec{\lambda})\,\vec{f}(x,t)\cdot\vec{\lambda}(t)\ B(x,\vec{\lambda})\right]|\ 0>$$

$$= <0\ |\mathrm{T}{:}\exp\left[\int dx\int d\rho(\vec{\lambda})\,B^{\dagger}(x,\vec{\lambda})\,\{\exp[\int dt\,\frac{i}{\hbar}\vec{f}(x,t)\cdot\vec{\lambda}(t)] - 1\}\,B(x,\vec{\lambda})\right]{:}|\ 0>$$

$$= <0\ |{:}\exp\left[\int dx\int d\rho(\vec{\lambda})\,B^{\dagger}(x,\vec{\lambda})\,\{\mathrm{T}\exp[\int dt\,\frac{i}{\hbar}\vec{f}(x,t)\cdot\vec{\lambda}(t)] - 1\}\,B(x,\vec{\lambda})\right]{:}|\ 0>$$

$$= \exp\left[-\int dx\int d\rho(\vec{\lambda})\ \{1 - \mathrm{T}\exp[\int dt\ \frac{i}{\hbar}\vec{f}(x,t)\cdot\vec{\lambda}(t)\ ]\}\right]$$

$$= \exp\left[-\int dx\int d\rho(\vec{\lambda})\ \{1 - \mathrm{T}\exp[\int dt\,\frac{i}{\hbar}\vec{f}(x,t)\cdot\vec{\lambda} - \int dt\,h(\vec{\nabla}_{\lambda},\vec{\lambda})\ ]\}\right]$$

where $h(\vec{\nabla}_{\lambda},\vec{\lambda})\cdot 1 = 0$ and $d\rho(\vec{\lambda}) = g\frac{1}{\lambda^{\mu}}\exp\left\{-\frac{(mb)}{\hbar}\lambda^{2} - 2y(\lambda,b,\hbar)\right\}d\lambda d\Omega,\ 1 \le \mu < 3.$

It would be interesting to try to reformulate this final expression by means of a path integral, but so far we have not succeeded in doing so. However, we can still predict that the result is nontrivial, since we have already obtained the nontrivial characteristic functional of infinite-component fields as the limit of finite-component ones, see (6.35) or (6.41).

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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